

Independent Randomization Devices and the Elicitation of Ambiguity Averse Preferences

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Abstract

Random incentive mechanisms are an important tool of experimental economics. According to such mechanisms subjects are asked to choose from a variety of problems, but are paid according to only one - randomly drawn - problem. I show that subjects' choices in random incentive mechanisms and single choice experiments are identical if they view the randomization device that determines the choice problem as "stochastically independent". This behavioral notion of stochastic independence is shown to be compatible with the maxmin expected utility model *as well as* with the smooth ambiguity averse model. This stands in contrast with one of the main streaks of the debate surrounding these two models, which claims that stochastic independence is compatible with the former model but not with the latter. These results can be viewed as a counterpoint to the observation that random incentive mechanisms can only be used to elicit preferences over objective lotteries when these preferences satisfy the independence axiom, the present article shows that this observation does not extend to an environment of subjective uncertainty.

Random Incentive (RI-) Mechanisms are widely used in Experimental Economics. In such mechanisms subjects are presented with a variety of choice problems. They are asked

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to make a choice from each problem. However, their payment depends only on one of these choices, which is randomly drawn.

RI-mechanisms are an alternative to performing separate experiments for every single choice problem. If the subjects' behavior in the separate single choice experiments is identical to the subjects' behavior in the RI-mechanism, the RI-mechanism is said to satisfy *isolation*. In that case, the subject acts as if he was dealing with every choice problem in isolation. If isolation is satisfied RI-mechanisms have many advantages over separate single choice experiments. Most importantly the subject has the same endowment for each choice problem. This rules out income effects as well as effects through the substitutability or complementarity among the choices from the different problems. If isolation does not hold, it is not clear how one should interpret the data generated by RI-mechanisms.

Becker Marshak de Groot (BdGM-) and Portfolio Choice (PC-) mechanisms are prominent examples of RI-mechanisms. In BdGM-mechanisms agents evaluate some object, often an objective lottery or a subjectively uncertain payment plan, through a set of binary choice problems, each of which consisting of that object a non-random payment.¹ If isolation holds agents choose the object in each of the problems with a payment that is lower than the value of the object, otherwise they choose the payment. The choice sets of PC-mechanisms are budget sets.² The advantage of PC-mechanisms over BdGM-mechanisms is that the choice sets of PC-mechanisms are - much - larger. While an agent reveals on statement of preference per choice problem in a BdGM mechanism, namely whether he prefers the object to the payment or not, the agent reveals a plentitude of such statements in PC-mechanisms, namely that he prefers his choice to any other available bundle in the given budget set.³

Both BdGM- and PC-mechanisms have been used in experiments on ambiguity aversion. Many of the experiments reviewed by Camerer and Weber [7] use BdGM-mechanisms to elicit certainty equivalents for some ambiguous acts. More recently Halevey [19] ap-

¹BdGM-mechanism where defined by Becker de Groot and Marshak [3]. They are used extensively.

²PC-mechanisms are an invention of Choi et al. [6].

³Choi et al. [6] created portfolio choice experiments to generate much larger data sets than BdGM-mechanism using approximately the same amount of resources (time and money). In each problem the agent faces a large set of choices. If isolation holds each of the agent's choices can be interpreted as the revealed preference of that choice over all other available options.

plied the data generated using a BdGM-mechanism towards a discussion of the relative merits of different representations of ambiguity averse preference. Ahn et al. [1] use a PC-mechanism to distinguish the explicatory power of different representations of ambiguity averse preferences. However, there are no results on the theoretical validity of RI-mechanisms in an environment of ambiguity averse agents. It is now known whether ambiguity aversion is compatible with isolation holding for RI-mechanisms. The present study argues that the answer to this question revolves around the notion of stochastic independence. To gain some intuition for this consider the following stark example of an RI-mechanism:

There are six choice problems, a dice is rolled to determine the payoff relevant problem. Assume that each choice problem consists in making a choice of a set of payment plans which are conditioned on some events A, B and C . If a subject is an expected utility maximizer, then the set of all such RI-mechanisms satisfies isolation if and only if the events A, B and C are stochastically independent of the die. If not, that is if A was, for example, more probable when conditioning on the die coming up at one than when conditioning on the die coming up at two, then the subjects preferences over payments plans should depend on the roll of the die.

So we might start out with the hypothesis that RI-mechanisms satisfy isolation if and only if stochastically independent randomization devices are used. It is straightforward to check this hypothesis when subjects are expected utility maximizers (I do so in Lemma 1). Intuitively the same condition should apply to the case in which the subject considers the events A, B and C as ambiguous. However, with ambiguity averse agents, this hypothesis is anything but straightforward to check, as we do not have an agreed upon definition of stochastic independence for that case. The standard definition of stochastic independence cannot be used since this definition would require the agent to hold *one* probability measure on the state space. However, ambiguity aversion is usually modeled in a way that stands in contradiction with this assumption.

Matters are further complicated by the apparent belief that some of the preminent models of ambiguity aversion are incompatible with the notion of stochastic independence. In this vein, Epstein [10] argues that the smooth ambiguity averse representation by Klibanoff et. al. [27] is incompatible with the existence of stochastically independent ambiguous events. Interestingly, Klibanoff et. al. [28] do not take issue with this argu-

ment of Epstein. They only disagree with Epstein’s evaluation of the argument: while Epstein [10] argues that a model of ambiguity averse preferences should allow for stochastic independence, Kilbanoff et. al. [28] argue that it is an advantage of their model that stochastically independent events as envisioned by Epstein [10] are ruled out.

As a first contribution I define a behavioral notion of stochastic independence, which collapses to standard stochastic independence when restricting attention to expected utility maximizers, and show that any RI-mechanism which uses a randomization device that is independent according to this notion satisfies isolation (Theorem 1). The second contribution lies in showing that, contrary to the conjectures derived from Epstein [10] and Klibanoff et. al. [28], both the model of maxmin expected utilities and the smooth model of ambiguity averse preferences are compatible with stochastic independence - at least with the kind of stochastic independence necessary for RI-mechanisms to satisfy isolation (Theorem 2). In sum, the second contribution of the present paper consists in showing that the preeminent representation models for ambiguity averse preferences can be formulated such that RI-mechanisms satisfy isolation.

Standard (probabilistic) stochastic independence is a symmetric relation: some event E is independent of another event G if and only if G is independent of E . The behavioral notion used in the prior two theorems has no a priori requirement for such symmetry to hold. Indeed for RI-mechanisms to satisfy isolation it is important that the agents preferences over their choices do not depend on the draws from the randomization device. Conversely, it is unproblematic for an agents preferences over acts that are conditioned on the randomization device to depend on the outcomes of some of the choices in the separate problems - the agent simply never has to evaluate any such act. Since the behavioral notion of stochastic independence coincides with the probabilistic notion when restricting attention to expected utility maximizers, the symmetry of the independence relation is guaranteed in that environment. I show that this symmetry *cannot* hold with ambiguity averse agents whose preference can be represented by one of the two models. The observation that stochastic independence is necessarily an asymmetric property in the given context is the third contribution (Theorem 3).

The fourth contribution consists in a way out of this dilemma, which is to demand that only a subset of all RI-mechanisms should satisfy isolation. The set of all BdGM-mechanisms satisfies isolation if and only if a much weaker behavioral condition of stochas-

tic independence applies to the randomization device. I go on to show that this weaker notion of stochastic independence permits mutually independent ambiguous events when the agents are maxmin expected utility maximizers. If we require the notions of independence to be symmetric and restrict attention to BdGM-mechanisms, there are specifications of maxmin expected utilities such that all BdGM-mechanisms satisfy isolation. Unfortunately, this same solution does not apply to the case of preferences that are representable via the smooth model of ambiguity aversion: for these preferences symmetry of independence and ambiguity aversion are mutually exclusive - whether one uses the stronger or the weaker notion of stochastic independence (Theorem 4).

1 Preliminaries

1.1 Preferences

Preferences are defined over a space of Anscombe-Aumann acts. There is a separable metric state space Ω , and a space of simple lotteries, $\Delta(X)$, over some outcomes X , where X is an interval of real numbers with 0 in its interior. The set of probability measures on Ω is denoted by $\Delta(\Omega)$. Preferences are defined over the space of all acts $f : \Omega \rightarrow \Delta(X)$. A constant act assigns to every state the same consequence $p \in \Delta(X)$, so $f(\omega) = p$ for all $\omega \in \Omega$. As a shorthand such constant acts are directly denoted by p . If p in turn, is such that $p(x) = 1$ for some consequence $x \in X$, the act p is denoted by x .

For any set of acts (f_1, \dots, f_n) and any partition (E_1, \dots, E_n) of Ω , the compound act $(E_1 : f_1; \dots; E_n : f_n) : \Omega \rightarrow \Delta(X)$ is defined by $(E_1 : f_1; \dots; E_n : f_n)(\omega) = f_i(\omega)$ for $\omega \in E_i$. The act $(E_1 : f_1; \dots; E_n : f_n)$ is called a σ/σ' -compound act if $E_i \in \sigma'$ and f_i a σ -measurable act for all i . Mixtures of acts $\alpha f + (1 - \alpha)g$ are defined componentwise so $(\alpha f + (1 - \alpha)g)(\omega) = \alpha f(\omega) + (1 - \alpha)g(\omega)$ for all $\omega \in \Omega$. An element $x_f \in X$ is called the certainty equivalent of f if $x_f \sim f$. I only consider preferences that obey the following four axioms:

(MON): Take any two acts f, g . If $f(\omega) \succeq g(\omega)$ for all ω then $f \succeq g$, if $f(\omega) \succ g(\omega)$ for all ω then $f \succ g$.

(L-IND) Take any three constant acts p, q, r . If $p \succeq q$ then $\alpha p + (1 - \alpha)r \succeq \alpha q + (1 - \alpha)r$ for all $\alpha \in (0, 1)$.

(UA) Take any two acts f, g . If $f \sim g$ then $\alpha f + (1 - \alpha)g \succsim f$ for all $\alpha \in [0, 1]$.

(CON) Take any three acts f, g, h . If $f \succ g$ and $g \succ h$ then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha)h \succ g$ and $g \succ \beta f + (1 - \beta)h$.

The axiom (MON) is a requirement of monotonicity (or state independence), it says that an agent should prefer f to g if he prefers f to g in every state. The axiom (L-IND) applies the independence axiom to the subspace of constant acts (lotteries), it states that if one (objective) lottery p is preferred to another q , then a mixture of p with some other lottery r should be preferred to a mixture of q with r in the same proportions. Without (MON) and (L-IND) isolation would not even hold for BdGM-mechanisms involving only objective lotteries.⁴ Consequently the first two axioms are necessary, to derive any novel results for the context of ambiguity aversion. The third axiom (UA) is Schmeidler’s [37] uncertainty aversion axiom. For preferences that satisfy this axiom an objective mixture over two indifferent acts must be weakly preferred to either one of the acts. This axiom weakens the independence axiom of Anscombe and Aumann (IND) which extends the requirement of (L-IND) from the set of constant acts to the set of all acts, formally, according to (IND), for any three acts f, g, h and any $\alpha \in (0, 1)$, the preference $f \succ g$ implies the preference $\alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$. Finally for technical reasons I only study preferences that satisfy a form of the continuity axiom (CON).

1.2 Utility Representations

Gilboa and Schmeidler’s [16] maxmin expected utilities and Klibanoff, Marinacci and Mukherji’s [27] smooth model of ambiguity aversion are consistent with (MON), (L-IND), (UA) and (CON). I chose to frame some of the main results of the present article in terms of these two representations, since they play a major role in applications and experiments. One of the main goals of Halevy [19] and of Ahn et al. [1] is to distinguish the explicatory power of these two models. Moreover, they are special cases of a large set of different representations of ambiguity averse preferences.⁵ Therefore the main existence result in

⁴This observation is due to Karni and Safra [23] as well as Holt [22]. I will place the results of the present paper into the context of the literature on this observation in Section 7

⁵Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [5] characterize the class of “uncertainty averse preferences” using only axioms of monotonicity, continuity and uncertainty aversion. This - large

the article, which is contained in Theorem 2 directly transfers to any of the superclasses of the named two representations.

Definition 1 *A maxmin expected utility representation U of \succsim is defined through a Bernoulli utility functional $u : \Delta(X) \rightarrow \mathbb{R}$ and a convex and compact set of priors C on the state space Ω such that*

$$U(f) := \min_{\pi \in C} \int_{\Omega} u(f(\omega)) d\pi(\omega).$$

A smooth ambiguity averse representation V of \succsim is defined through Bernoulli utility functional $u : \Delta(X) \rightarrow \mathbb{R}$, a strictly concave and strictly increasing function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, and a measure μ over measures $\Delta(\Omega)$, such that

$$V(f) := \int_{\Delta(\Omega)} \phi \left(\int_{\Omega} u(f(\omega)) d\pi(\omega) \right) d\mu(\pi).$$

According to either definition the decision maker takes a set of priors on the state space Ω into account, C for the maxmin expected utility representation and $supp(\mu)$ for the smooth ambiguity averse representation. In the case of the first representation the agent evaluates the utility of an act f as the minimal expected utility attributed to this act according to the set of priors C . According to the second representation the agent calculates the expected utility of f for every prior $\pi \in supp(\mu)$ and then calculates his overall utility as the weighed average of a concave transformation ϕ of all these utilities, with μ determining the weight of each prior π . In either case an expected utility representation arises for C or $supp(\mu)$ being a singleton. The smooth ambiguity averse representation implies a particular timing of nature's actions, in a first stage, she picks a distribution π in and then draws a state ω from that distribution π in the second stage. For any given probability measure π on Ω let π_{σ} denote the marginal of π with respect to the algebra σ .

- class of preferences comprises the set of all preferences representable through either a maxmin expected utility or by the smooth model of ambiguity aversion. The set of variational preferences, which have been characterized by Maccheroni, Marinacci, and Rusticchini [35] is nested between the set of uncertainty averse preferences, as characterized by Cerreia et al. [5] and the set of preferences representable through maxmin expected utilities. The set of preferences representable through an α -maxmin expected utility, as characterized by Ghirardato, Maccheroni and Marinacci [15] does constitute a superclass of maxmin expected utilities without being a subclass of the class of uncertainty averse preferences.

1.3 Random Incentive Mechanisms

I now provide formal definitions of RI-, BdGM- and PC-mechanisms. A σ/σ' -**RI-mechanism** $(\mathcal{E}, \mathcal{S})$ consists in a finite partition of Ω : $\mathcal{E} = (E_1, \dots, E_n)$ with $E_i \in \sigma'$ for all i and a set of equally many choice sets $\mathcal{S} = (S_1 \times \dots \times S_n)$, with all $f_i \in S_i$ being σ -measurable. The algebra σ' is also called a **randomization device**. The task of the subject is to choose an act f_i out of every set S_i . His payment is then determined by the σ/σ' -compound act $(E_1 : f_1; \dots; E_n : f_n)$. So the maximization problem of the agent can be described as follows:

$$\begin{aligned} & \text{choose } (f_1^*, f_2^*, \dots, f_n^*) \in \mathcal{S} \\ & \text{such that} \\ & (E_1 : f_1^*; E_2 : f_2^*; \dots; E_n : f_n^*) \succsim (E_1 : f_1; E_2 : f_2; \dots; E_n : f_n) \\ & \text{for all } (f_1, f_2, \dots, f_n) \in \mathcal{S}. \end{aligned}$$

A RI-mechanism is said to satisfy **isolation** if $f_i^* \succsim f_i$ for all $i = 1, \dots, n$. It is the goal of the present article to identify conditions on the relation between the algebras σ and σ' such that all σ/σ' -RI-mechanisms satisfy isolation.

BdGM-mechanisms constitute examples of RI mechanisms. A σ/σ' -RI-mechanism $(\mathcal{E}, \mathcal{S})$ is considered a σ/σ' -BdGM mechanism if $S_i = \{f, x_i\}$, for all $i = 1, \dots, n$. As a shorthand any σ/σ' -BdGM mechanism can be denoted as (f, g) instead of $(\mathcal{E}, \mathcal{S})$ with the understanding that g is a σ' -measurable act with a finite image set $g(\Omega) = \{x_1, \dots, x_n\}$, $E_i := g^{-1}(x_i)$ and $S_i = \{f, x_i\}$. Given that preferences are presumed to be monotonic, the set of all σ/σ' -BdGM-mechanisms satisfies isolation if and only if the subjects should choose x_i in the separate decision problems if $x_i > x_f$ and f otherwise.

A **PC-mechanism** following Choi et al. [6] and Ahn et al. [1] is a σ/σ' -RI-mechanism $(\mathcal{E}, \mathcal{S})$ with the feature that \mathcal{S} is a set of budget sets consisting of various assets: a_1, \dots, a_k . Each asset is a σ -measurable act $a_j : \Omega \rightarrow \Delta(X)$. An agent who owns a bundle $\mathbf{q} : (q_1, \dots, q_k)$, where q_j denotes the quantity of asset a_j in the agents portfolio, receives the aggregate payoff $f_{\mathbf{q}}$, with $f_{\mathbf{q}}(\omega) := \sum_{j=1}^k q_j a_j(\omega)$ for all $\omega \in \Omega$. As a weighed sum of σ -measurable acts $f_{\mathbf{q}}$ is itself σ -measurable. The budget set S_i is fully characterized by the agents income I^i and the prices of the assets p_j^i for all $1 \leq j \leq k$. It can be represented as $S_i := \{f_{\mathbf{q}} : \sum_{j=1}^k p_j^i q_j \leq I^i\}$.

2 Examples of RI-mechanisms

Throughout this section the algebra σ' is generated by the partition (E, \bar{E}) and σ by the partition (A, B, C) .

Example 1 Ahn et al. [1] use the experimental data generated by a PC-mechanism to compare the explicatory power of various models of decision making under ambiguity. Here, I show that isolation can hold for a stylized version their mechanism with ambiguity averse agents whose preferences can be represented by the smooth model. To this end, define a PC-mechanism $(\mathcal{E}, \mathcal{S})$ with three assets a_A, a_B and a_C , each of which pays 1 in the event A, B and C respectively and 0 in the respective complementary event \bar{A}, \bar{B} and \bar{C} . Let and portfolio of these assets be denoted by $\mathbf{q} = (q_A, q_B, q_C)$ such that $f_{\mathbf{q}}(\omega) = q_A$ for $\omega \in A$, $f_{\mathbf{q}}(\omega) = q_B$ for $\omega \in B$ and $f_{\mathbf{q}}(\omega) = q_C$ for $\omega \in C$. The randomization device \mathcal{E} is such that $E_1 = E$, $E_2 = \bar{E}$; the two budget sets are defined as follows: $S_1 = \{f_{\mathbf{q}} \mid \frac{q_A}{50} + \frac{q_B}{90} + \frac{q_C}{30} = 1\}$ and $S_2 = \{f_{\mathbf{q}} \mid \frac{q_A}{50} + \frac{q_B}{30} + \frac{q_C}{90} = 1\}$.⁶

Ahn et al. [1] revealed to their subjects that A obtains with probability $\frac{1}{3}$; no (further) information was provided on the other two events. The following smooth ambiguity averse representation $V(f) = \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega)) d\pi(\omega)\right) d\mu(\pi)$ is consistent with this setup: $\phi = \ln$ and $\text{supp}(\mu) = \{\pi^1, \pi^2, \pi^3, \pi^4\}$ with each of the measures being equally likely according to the second order probability μ , that is $\mu(\pi^i) = \frac{1}{4}$ for $i = 1, 2, 3, 4$. In addition let

$$\begin{aligned}\pi^1(E) &= \pi^2(E) = \pi^3(\bar{E}) = \pi^4(\bar{E}) = 1 \\ \pi^1(A) &= \pi^2(A) = \pi^3(A) = \pi^4(A) = \frac{1}{3} \\ \pi^1(B) &= \pi^2(C) = \pi^3(B) = \pi^4(C) = \frac{1}{9}.\end{aligned}$$

Only two marginals with respect to σ arise with positive probability: π_{σ}^* and π_{σ}° . In accordance with the assumption that the subject knows the probability of A to be $\frac{1}{3}$; we have that $\pi_{\sigma}^*(A) = \pi_{\sigma}^{\circ}(A) = \frac{1}{3}$. Conversely, to reflect on the ambiguity with respect to the two other events the π_{σ}^* and π_{σ}° assign different probabilities to these:

⁶This experimental setup deviates from Ahn et al. [1]'s in a number of more or less relevant ways. In their paper agents have to make 50 instead of 2 choices, budget sets are randomly drawn and probably most importantly: at the time that an agent has to choose from some budget set he does not yet know the next budget set. This last difference introduces yet another layer of uncertainty into the problem, which I do not analyze in the present paper.

$\pi_\sigma^*(B) = \frac{1}{9} \neq \pi_\sigma^\circ(B) = \frac{5}{9}$ and $\pi_\sigma^*(C) = \frac{5}{9} \neq \pi_\sigma^\circ(C) = \frac{1}{9}$. The subject is an expected utility maximizer with respect to E , he believes that E occurs with probability one half and that nature determines whether E or \bar{E} occurs in the first stage. Next, the subject's beliefs on the events in σ do not depend on whether E occurred or not: the measure μ ascribes a probability of one half each to π_σ^* and to π_σ° , whether the agent conditions on the event $\{\pi \mid \pi(E) = 1\}$ or on the event $\{\pi \mid \pi(\bar{E}) = 1\}$ or does not condition at all. Now to see that the given PC-mechanism satisfies isolation for V , choose any bundles $\mathbf{q} \in S_1$ and $\mathbf{q}' \in S_2$ and observe that,

$$\begin{aligned}
V((E : f_{\mathbf{q}}; \bar{E} : f_{\mathbf{q}'})) &= \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u((E : f_{\mathbf{q}}; \bar{E} : f_{\mathbf{q}'})(\omega))d\pi(\omega)\right)d\mu(\pi) = \\
\mu(\{\pi \mid \pi(E) = 1\}) &\int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f_{\mathbf{q}}(\omega))d\pi(\omega)\right)d\mu(\pi \mid \{\pi \mid \pi(E) = 1\}) + \\
\mu(\{\pi \mid \pi(E) = 0\}) &\int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f_{\mathbf{q}'}(\omega))d\pi(\omega)\right)d\mu(\pi \mid \{\pi \mid \pi(E) = 0\}) = \\
\mu(\{\pi \mid \pi(E) = 1\}) &\int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f_{\mathbf{q}}(\omega))d\pi(\omega)\right)d\mu(\pi) + \\
\mu(\{\pi \mid \pi(E) = 0\}) &\int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f_{\mathbf{q}'}(\omega))d\pi(\omega)\right)d\mu(\pi) = \\
\mu(\{\pi \mid \pi(E) = 1\}) &V(f_{\mathbf{q}}) + \mu(\{\pi \mid \pi(E) = 0\})V(f_{\mathbf{q}'}).
\end{aligned}$$

The first equality follows by definition, the second splits the integral over all measures π into two parts: first an integral over the measures with $\pi(E) = 1$ then an integral over the remainder, and restricts attention to the parts of the compound act $(E : f_{\mathbf{q}}; \bar{E} : f_{\mathbf{q}'})$ that matter in each of the two cases. The third equality relies on the observation that the measure μ on the marginals π_σ^* and π_σ° does not depend on the conditioning events $\{\pi \mid \pi(E) = 1\}$ and $\{\pi \mid \pi(E) = 0\}$. So $V((E : f_{\mathbf{q}}; \bar{E} : f_{\mathbf{q}'}))$ is maximized if and only if $V(f_{\mathbf{q}})$ and $V(f_{\mathbf{q}'})$ are maximized; isolation holds.

Now observe that very few of the particulars of $(\mathcal{E}, \mathcal{S})$ and C were used to draw this conclusion. It did not matter that only budget sets were considered, the arguments directly extend to any σ/σ' -RI-mechanism. Moreover, the particular definition of μ was not used, all that mattered was that, on the one hand, the measure over marginals on σ does not depend on any conditioning event $E' \in \sigma'$ and, on the other hand, that $\pi(E') = 0$ or $\pi(E') = 1$ for all $\pi \in \text{supp}(\mu)$. These observations are being generalized in Theorem 2.

Example 2 Reconsider the PC-mechanism defined in the prior example, but assume that the preferences of agents can be represented by a maxmin expected utility function $U(f) = \min_{\pi \in C} \int_{\Omega} u(f(\omega)) d\pi(\omega)$ where C is defined as the convex hull of the four priors π^5, π^6, π^7 and π^8 which are defined through the following criteria: According to each prior the algebra's σ and σ' are stochastically independent. Moreover, $\pi^5(E) = \pi^6(E) = \frac{1}{4}$, $\pi^7(E) = \pi^8(E) = \frac{3}{4}$, $\pi_{\sigma}^5 = \pi_{\sigma}^7 = \pi_{\sigma}^*$, and $\pi_{\sigma}^6 = \pi_{\sigma}^8 = \pi_{\sigma}^{\circ}$, with π_{σ}^* and π_{σ}° being defined as above. The algebras σ and σ' would be considered stochastically independent according to the notions of stochastic independence for maxmin expected utilities by Gilboa and Schmeidler [16], Klibanoff [26]⁷, and Epstein [10]. Moreover, as required by the experimental setup of Ahn et al. [1] according to any prior in the set C subjects assign a probability of $\frac{1}{3}$ to A occurring. They are uncertain about the probability of the other two events.

With this new specification of preferences the PC-mechanism does not satisfy isolation, as agents would choose the bundle $(50, 0, 0)$ from the two budget sets in single choice experiments, whereas choosing $(0, 90, 0)$ and $(0, 0, 90)$ from S_1 and S_2 would make them better off when playing the PC-mechanism. To see that $(50, 0, 0)$ is the optimal choice in S_1 , consider the tradeoff between investing in a_A or the two other assets. Suppose the agent decreases his spending on a_A by some $\epsilon > 0$. Since $U(f_{\mathbf{q}}) = \min_{\pi_{\sigma} \in \text{co}\{\pi_{\sigma}^*, \pi_{\sigma}^{\circ}\}} (q_A \pi_{\sigma}(A) + q_B \pi_{\sigma}(B) + q_C \pi_{\sigma}(C)) = \min \left\{ \frac{q_A}{3} + \frac{q_B}{9} + \frac{5q_C}{9}, \frac{q_A}{3} + \frac{5q_B}{9} + \frac{q_C}{9} \right\}$, it is optimal to allocate the freed up resources $\frac{\epsilon}{50}$ equally on a_B and a_C . Defining the increase of the investment in each asset as δ we obtain that $\frac{\delta}{90} + \frac{\delta}{30} = \frac{\epsilon}{50}$, which implies in turn that $\delta = \frac{45\epsilon}{100}$. Now observe that reducing q_A by ϵ decreases the utility by $\frac{\epsilon}{3}$, whereas increasing q_B and q_C by $\frac{45\epsilon}{100}$ each increases the utility by $\frac{45\epsilon}{100} (\frac{1}{9} + \frac{5}{9}) = \frac{3\epsilon}{10} < \frac{\epsilon}{3}$. It is therefore optimal to only invest in asset a_A , where $U(f_{(50,0,0)}) = \frac{50}{3}$.

However in the RI-mechanism it is better for the subject to choose $(E : f_{(0,90,0)}, \bar{E} :$

⁷Klibanoff's notion of a stochastically independent randomization device also requires that any act defined with respect to the randomization device should have an expected utility representation. This is not required by the present definition.

$f_{(0,0,90)}$) which yields a utility of:

$$\begin{aligned}
& U((E : f_{(0,90,0)}, \bar{E} : f_{(0,0,90)})) = \\
& \min_{\pi \in C} \int_{\Omega} u((E : f_{(0,90,0)}, \bar{E} : f_{(0,0,90)})(\omega)) d\pi(\omega) = \\
& \min_{x \in [\frac{1}{4}, \frac{3}{4}], y \in [\frac{1}{9}, \frac{5}{9}]} 90(xy + (1-x)(\frac{2}{3} - y)) = \\
& 20 > \frac{50}{3} = U((E : f_{(50,0,0)}, \bar{E} : f_{(50,0,0)}))
\end{aligned}$$

So for the given set of beliefs C the portfolio choice experiment does not satisfy isolation - even if the randomization device is “stochastically independent” according to a range of different criteria. According to the given belief set, the subject can hedge between the two problems, in the sense that both need to be evaluated with the same marginal π_{σ} .

Example 3 It turns out that any BdGM-mechanisms with the same specification of preferences as in the prior example does satisfy isolation. Define a σ/σ' -BdGM-mechanisms (f, g) with $g(\omega) = x$ for $\omega \in E$ and $g(\omega) = y$ otherwise. To see that (f, g) satisfies isolation observe that:

$$\begin{aligned}
U((E : x; \bar{E} : f)) &= \min_{\pi \in C} \int_{\Omega} u((E : x; \bar{E} : f)(\omega)) d\pi(\omega) = \\
& \min_{\rho \in [\frac{1}{4}, \frac{3}{4}], \pi_{\sigma} \in \text{co}(\{\pi_{\sigma}^*, \pi_{\sigma}^{\circ}\})} \rho u(x) + (1 - \rho) \int_{\Omega} u(f(\omega)) d\pi_{\sigma}(\omega) = \\
& \min_{\rho \in [\frac{1}{4}, \frac{3}{4}]} \rho u(x) + (1 - \rho) U(f) = U((E : x; \bar{E} : x_f)) \\
& \text{and} \\
U((E : f; \bar{E} : x_f)) &= \min_{\pi \in C} \int_{\Omega} u((E : f; \bar{E} : x_f)(\omega)) d\pi(\omega) = \\
& \min_{\rho \in [\frac{1}{4}, \frac{3}{4}], \pi_{\sigma} \in \text{co}(\{\pi_{\sigma}^*, \pi_{\sigma}^{\circ}\})} \rho \int_{\Omega} u(f(\omega)) d\pi_{\sigma}(\omega) u(x) + (1 - \rho) u(x_f) = U(f).
\end{aligned}$$

So the agent chooses f out of $S_2 = \{f, y\}$ if and only if $f \succsim y$. The same argument applies to the agents choice out of the first set $S_1 = \{x, f\}$. In short, every BdGM mechanism (f, g) satisfies isolation. Observe that this example can easily be generalized as no particular characteristics of the set of marginals on σ did play any role in the argument.

3 Stochastic Independence

The first main contribution of the paper consists in giving behavioral conditions of stochastic independence under which RI-mechanisms in general and BdGM-mechanisms in particular satisfy isolation. Before doing so, let me state that for an expected utility maximizer the set of all σ/σ' -RI-mechanisms satisfies isolation if and only if σ and σ' are stochastically independent. The proof of Lemma 1, just as all other proofs that do not appear in the text, can be found in the Appendix.

Lemma 1 *Assume that subjects are expected utility maximizers. The following three conditions are equivalent.*

- (i) *The subjects view σ and σ' as stochastically independent.*
- (ii) *The set of all σ/σ' -RI mechanisms satisfies isolation.*
- (iii) *The set of all σ/σ' -BdGM mechanisms satisfies isolation.*

Does a version of Lemma 1 hold for subjects with ambiguity averse preferences? There is no straightforward answer to this question, given that the standard notion of independence does not apply to the case of ambiguity averse agents. To fill this question with meaning let me next give some behavioral notions of stochastic independence that can be applied to ambiguity averse preferences. The common idea behind the following two notions of stochastic independence goes as follows: an algebra σ is said to be stochastically independent of another algebra σ' according to the preferences \succsim if the agent is indifferent between playing some σ -measurable act f if some $E \in \sigma'$ occurs and obtaining the (unconditional) certainty equivalent of f if E occurs.

Definition 2 *An algebra σ is **weakly independent** of another algebra σ' if*

$$(E : f; \bar{E} : g) \sim (E : x_f, \bar{E} : g) \text{ holds for all } \sigma - \text{measurable acts } f, \\ \text{all } \sigma' - \text{measurable acts } g, \text{ and all } E \in \sigma'.$$

*An algebra σ is **strongly independent** of another algebra σ' if*

$$(E_1 : f_1; E_2 : f_2; \dots; E_n : f_n) \sim (E_1 : x_{f_1}; E_2 : f_2; \dots; E_n : f_n) \\ \text{holds for all } \sigma/\sigma' - \text{compound acts } (E_1 : f_1; E_2 : f_2; \dots; E_n : f_n).$$

Note that both definitions rely on the idea that σ is stochastically independent of σ' if the subjects like or dislike of some σ -measurable act f does not depend on some event $E \in \sigma'$ occurring or not. The definitions only differ in applying this condition of indifference to smaller or larger sets of cases: If the algebra σ is strongly independent of the algebra σ' , then it is also weakly independent of σ' , since any σ' -measurable act g can be represented as a σ/σ' -compound act $(E_1 : f_1; E_2 : f_2; \dots; E_n : f_n)$ with all f_i being constant acts. Klibanoff's [26] behavioral notion of stochastic independence applies the same condition to a yet smaller set of compound acts, I'll explicitly define and make use of his definition in the Appendix.⁸

If an agents' preferences are representable through an expected utility function, the two notions of stochastic independence coincide with the standard notion of stochastic independence.⁹ Conversely since each of the two definitions requires that the agent separately reasons about the value of some σ -measurable acts in some event E , the definitions need not coincide for ambiguity averse agents. The phenomenon of ambiguity aversion can be interpreted as the refusal to perform such mental separation; in the two named representations ambiguity averse agents attribute more probability mass to events with bad outcomes. Bearing this in mind the condition of strong independence appears as an utterly demanding requirement. The sure thing principle has to hold for a large subset of the set of all statements of preferences.

It therefore comes as some surprise that strong independence and ambiguity aversion are not incompatible. To see this observe that σ is strongly independent of σ' according to the smooth ambiguity averse representation given in Example 1: Extend the arguments leading up to the equality $V((E : f_{\mathbf{q}}; \bar{E} : f_{\mathbf{q}'})) = \mu(\{\pi \mid \pi(E) = 1\})V(f_{\mathbf{q}}) + \mu(\{\pi \mid \pi(E) = 0\})V(f_{\mathbf{q}'})$ to the case in which $f_{\mathbf{q}}$ and $f_{\mathbf{q}'}$ are replaced by any two σ -measurable acts f, f' , yielding $V((E : f; \bar{E} : f')) = V((E : x_f; \bar{E} : x_{f'}))$.

According to the function U defined in Example 2 σ weakly but not strongly independent of σ' . Observe that $U((E : f; \bar{E} : x)) = U((E : x_f; \bar{E} : x))$ and $U((E : x; \bar{E} : f)) = U((E : x; \bar{E} : x_f))$ holds for all σ -measurable acts f , not just for acts in S_1 and S_2 . So σ is weakly independent of σ' for the given maxmin expected utility U . To see that σ is not strongly independent of σ' remember that the most important argument

⁸The only other behavioral notion of stochastic independence I am aware of is the one by Epstein [10].

⁹The proof is very similar to the proof of Lemma 1. It is omitted here since the result can also be derived as a direct consequence of Lemma 1 and Theorem 1 which are both proved in this paper.

in the discussion of Example 2 was that $U((E : f_{(0,90,0)}, \bar{E} : f_{(0,0,90)})) > U(f_{(50,0,0)}) \geq U(f_{(0,90,0)}) = U(f_{(0,0,90)})$ holds. However strong stochastic independence would require that $U((E : f_{(0,90,0)}, \bar{E} : f_{(0,0,90)})) = U(f_{(0,90,0)}) = U(f_{(0,0,90)})$.

We are now ready to get back to the core of the article which is to validate the claim that RI-mechanisms work if they are using stochastically independent randomization devices. Theorem 1 can be read as an extension of Lemma 1 to the case of ambiguity averse agents.

Theorem 1 *The set of all σ/σ' -RI-mechanisms satisfies isolation, if and only if σ is strongly independent of σ' . The set of all σ/σ' -BdGM-mechanisms satisfies isolation if σ is weakly independent of σ' .*

The first part of Theorem 1 provides a necessary and sufficient condition under which any σ/σ' -RI-mechanism satisfies isolation. Since this condition is a very strong requirement of stochastic independence, the theorem contains a second part which relies on a significantly weaker notion of stochastic independence. Weak independence of the randomization device is sufficient for any BdGM-mechanism to satisfy isolation. A discussion of a necessary condition for all BdGM-mechanisms to satisfy isolation has been relegated to the Appendix. To understand stochastic independence within the two models of representation some more formalism is needed.

4 Some Formalism

Two algebras σ and σ' on Ω are considered stochastically independent according to π , if $\pi_{\sigma \cap \sigma'} = \pi_{\sigma} \times \pi_{\sigma'}$. Where $\sigma \cap \sigma'$ is defined as the algebra $\{H \mid H = E \cap G \text{ for some } E \in \sigma, G \in \sigma'\}$ and the product measure $\pi_{\sigma} \times \pi_{\sigma'}$ is defined through $(\pi_{\sigma} \times \pi_{\sigma'})(E \cap G) = \pi_{\sigma}(E)\pi_{\sigma'}(G)$ for all $E \in \sigma, G \in \sigma'$. The expression $\pi(\cdot \mid E)$ denotes the conditional probability with E as the conditioning event, $\pi_{\sigma}(\cdot \mid E)$ is the σ -marginal of that measure.

For a given set of beliefs C , the set C_{σ} denotes the set of all marginals π_{σ} of the beliefs in C , formally $C_{\sigma} = \{\pi_{\sigma} \mid \pi \in C\}$. The set $C_{\sigma} \times C_{\sigma'}$ is constructed as the convex hull of all product measures $\pi_{\sigma} \times \pi_{\sigma'}$, for all $\pi, \pi' \in C$, formally $C_{\sigma} \times C_{\sigma'} := \text{co}\{\pi_{\sigma} \times \pi_{\sigma'} \mid \pi, \pi' \in C\}$. The set $C_{\sigma}(\cdot \mid E)$ comprises the set of all σ -marginals conditional on E , formally $C_{\sigma}(\cdot \mid E) := \text{co}\{\pi_{\sigma}(\cdot \mid E) \mid \pi \in C\}$.

The algebra induced by the partition of $\Delta(\Omega)$ into sets of beliefs on Ω that share the same marginal on σ , that is the partition of $\Delta(\Omega)$ into sets of the form $[\pi^*] := \{\pi \mid \pi_\sigma = \pi_\sigma^*\}$, is denoted as $\bar{\sigma}$. The marginal of μ with respect to that algebra is denoted by $\mu_{\bar{\sigma}}$. So the smooth ambiguity averse utility of any σ -measurable act f , $V(f) = \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega))d\pi(\omega)\right)d\mu(\pi)$ can also be calculated as $\int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega))d\pi(\omega)\right)d\mu_{\bar{\sigma}}([\pi])$. The $\bar{\sigma}$ marginal of μ when conditioning on π belonging to some set Π is denoted by $\mu_{\bar{\sigma}}(\cdot \mid \Pi)$.

Armed with this terminology the utility function V given in Example 1 can now be alternatively defined through $\mu_{\bar{\sigma}}(\cdot \mid \{\pi \mid \pi(E) = 1\}) = \mu_{\bar{\sigma}}(\cdot \mid \{\pi \mid \pi(E) = 0\}) = \mu_{\bar{\sigma}}$, $\mu(\{\pi \mid \pi(E) = 1\}) = \mu\{\pi \mid \pi(E) = 0\} = \frac{1}{2}$, $\mu_{\bar{\sigma}}([\pi_\sigma^*]) = \mu_{\bar{\sigma}}([\pi_\sigma^\circ]) = \frac{1}{2}$ (π_σ^* and π_σ° are as defined above in Example 1). The set of beliefs C associated with the Utility function U defined in Example 2 can be represented much more succinctly as: $C = [\frac{1}{4}, \frac{3}{4}] \times co(\{\pi_\sigma^*, \pi_\sigma^\circ\})$.

Maxmin expected utilities and the smooth model of ambiguity averse preference both comprise the case of expected utility as a special case. Take any preferences \succsim that can be represented via either a maxmin expected utility or the smooth model of ambiguity aversion and the restriction of \succsim to some algebra σ : $\succsim|_\sigma$. These preferences $\succsim|_\sigma$ do have an expected utility representation if and only if (IND) holds for $\succsim|_\sigma$. In the case of the maxmin expected utility model (IND) holds for $\succsim|_\sigma$ if and only if C_σ is a singleton. In the case of the smooth model of ambiguity aversion (IND) holds for $\succsim|_\sigma$ if and only if either $supp(\mu_{\bar{\sigma}})$ is a singleton or if $\pi(E)\pi(\bar{E}) = 0$ holds for all $E \in \sigma$ and all $\pi \in supp(\mu)$. Note that I omit the commonly mentioned third condition for $\succsim|_\sigma$ to have an expected utility representation, namely that ϕ is linear. The reason for this being that my definition of the smooth ambiguity averse model assumes ϕ to be strictly convex.¹⁰ If $\succsim|_\sigma$ does not satisfy (IND) there exists an event $E \in \sigma$ such that $\succsim|_{\sigma^E}$ does not satisfy (IND), where σ^E is defined as the algebra $(\emptyset, E, \bar{E}, \Omega) \subset \sigma$. In this case the event E is called **ambiguous**.¹¹

¹⁰The requirement of strict concavity neatly unifies the requirement of ambiguity aversion (concavity) with that of smoothness - no kinks. In applications ϕ is generally chosen as strictly concave.

¹¹This definition coincides with Zhang's [40] definition of an ambiguous event. One issue in the debate around the smooth representation of ambiguity averse preferences is the question whether that model is able to differentiate between ambiguity and ambiguity attitude. Since Zhang's definition lumps these two concepts together, no differentiation between the two concepts arises in the current discussion. Epstein [10] argues that the smooth representation of ambiguity averse preferences cannot meaningfully differentiate between ambiguity and the attitude towards ambiguity.

5 Isolation and the two Models of Representation

The next theorem characterizes utility functions of both types for which σ is strongly independent of σ' .

Theorem 2 *Consider an algebra σ and another algebra σ' which is generated by the partition (E_1, \dots, E_n) . Assume that \succsim is representable through a maxmin expected utility (TeX smooth model of ambiguity aversion). Then σ is strongly independent of σ' if and only if Condition A (Condition B) holds, with*

$$(A) : \quad C = \left\{ \pi \mid \pi(H) = \sum_{i=1}^n \pi_{\sigma'}(E_i) \pi_{\sigma}^i(G_i) \right. \\ \left. \text{for any } H = \cup_{i=1}^n (E_i \cap G_i) \text{ with } G_i \in \sigma \text{ for all } i, \right. \\ \left. \pi_{\sigma'} \in C_{\sigma'}, \text{ and } \pi_{\sigma}^i \in C_{\sigma} \text{ for all } i \right\};$$

and

$$(B) : \quad \pi(E)(1 - \pi(E)) = 0 \text{ for all } \pi \in \text{supp}(\mu); E \in \sigma' \\ \mu_{\bar{\sigma}} = \mu_{\bar{\sigma}}(\cdot \mid \{\pi \mid \pi(E) = 0\}) \text{ for all } E \in \sigma'.$$

Note the similarity between condition (A) and Epstein and Schneider's [12] condition of rectangularity, which describes filtrations on which full Bayesian updating is dynamically consistent. If one replaces the the requirement that $\pi_{\sigma}^i \in C_{\sigma}$ with the requirement that $\pi^i \in C^i$ for all i the condition of rectangularity obtains. So there are two differences between (A) and rectangularity: Since rectangularity concerns the updating of *any* act, not just σ -measurable ones, the requirement $\pi^i \in C^i$ does not only concern the σ -marginals of beliefs. Rectangularity permits the sets of updates to vary with E_i , therefore the sets C^i are indexed by i . Conversely, as a condition of independence (A) demands that the sets of updates do not depend on learning any of the “independent” events E_i . The first part of Condition (B) requires that nature determines which cell of the partition (E_1, \dots, E_n) the state belongs to in the first stage of the compound lottery. The second part requires that all conditional distribution over σ -marginals $\mu_{\bar{\sigma}}(\cdot \mid \{\pi \mid \pi(E) = 1\})$ are identical for all events $E \in \sigma'$.¹²

¹²In a dynamic setting (B) would translate to there being no benefit from the cell of the partition (E_1, \dots, E_n) when choosing among σ -measurable acts.

Next note the similarity between the two conditions. Each one can be viewed as saying that when on conditions on any event $E \in \sigma'$ the agent should reason identically over the set of probability measures over σ . In the case of a maxmin expected utility this implies that the set of updates $C_\sigma(\cdot | E)$ are identical to C_σ for all $E \in \sigma'$; in the case of the smooth ambiguity averse model this implies that the measures over measures $\mu_{\bar{\sigma}}(\cdot | \{\pi | \pi(E) = 1\})$ are identical for all $E \in \sigma'$.

To see that conditions (A) and (B) are not vacuous, observe that the utility function V defined in Example 1 satisfies (B). Conversely, (A) does not hold for the maxmin utility U defined in Example 2. To see that some maxmin expected utilities actually satisfy (A) consider the convex hull of the set of all priors ψ for which the following holds: $\psi(E) \in \{\frac{1}{4}, \frac{3}{4}\}$, $\psi_\sigma(\cdot | E), \psi_\sigma(\cdot | \bar{E}) \in \{\pi_\sigma^*, \pi_\sigma^\circ\}$, with $\pi_\sigma^*, \pi_\sigma^\circ$ as defined above. Note that the set of priors C in Example 2 is a subset of this set, as it can be viewed as the convex hull of the set of priors ψ which in addition to the conditions mentioned above satisfies $\psi_\sigma(\cdot | E) = \psi_\sigma(\cdot | \bar{E})$. This set of priors was explicitly defined as $\{\pi^5, \pi^6, \pi^7, \pi^8\}$ in Example 2. So the present set of beliefs has more extrema, according to these additional extrema σ and σ' are not stochastically independent, as $\psi_\sigma(\cdot | E) \neq \psi_\sigma(\cdot | \bar{E})$ holds for each one of them. Theorem 2 provides the necessary theoretical backing to studies such as Ahn et. al [1] or Halevey [19] who did use random incentive mechanisms to distinguish between the explicatory power of these two models.

6 Stochastic Independence as a Symmetric Property

According to the probabilistic notion of stochastic independence an event E is independent of another event G if and only if G is independent of E . In that case stochastic independence is a symmetric relation. Note that the notions of stochastic independence defined here do not have an a priori symmetry requirement built in. The definitions are directional in the sense that they give criteria for some algebra σ to be (weakly or strongly) independent of σ' . A priori these criteria remain silent on the (weak or strong) independence of σ' of σ . Of course, when the underlying preferences have an expected utility representation, weak and strong independence coincide with the standard notion of stochastic independence. Consequently the independence relation is symmetric in that special case. In this section I show that symmetric strong independence and ambiguity

aversion are mutually exclusive within the two frameworks of preference representation discussed in the article.

Theorem 3 *Let an ambiguous algebra σ be strongly independent of another algebra σ' according to \succsim . Let \succsim be representable through either a maxmin expected utility or through the smooth model of ambiguity aversion. Then σ' cannot even be weakly independent of σ .*

Given the discussion of the notion of strong independence this result should not be too surprising. What is surprising is that strong independence of one algebra of another is compatible with the subjects ambiguity aversion. The fact that demanding the sure thing principle to hold for yet more cases implies that the given representations collapse to expected utility representations is to be expected.¹³ What is problematic though is that we usually frame stochastic independence as a symmetric property. So we might want to rule out any preferences that allow for asymmetric stochastic independence. If we use strong independence as the notion of stochastic independence Theorem 3 just leaves us the dire choice between either abandoning stochastic independence or ambiguity aversion.

In the following I show that things do not look as bad if one steps back from the grand project of finding preferences such that all RI-mechanisms satisfy isolation, but instead focusses on the class of most commonly used RI-mechanisms: BdGM-mechanisms. To state the next result it makes sense to say that some preferences \succsim satisfy ***independence as a symmetric relation*** (ISR) if for any two algebras σ and σ' , σ being weakly independent of σ' , implies that σ' is weakly independent of σ . In a nutshell an agent who satisfies this notion construes stochastic independence as a symmetric relation. It is

¹³Theorem 3 directly implies the following corollary about a weakened form of P2:

Corollary 1 *Let some preferences \succsim on all $\sigma \cap \sigma'$ -measurable acts be representable through a maxmin expected utility or by the smooth model of ambiguity aversion. Let $(E : f; \bar{E} : g) \succsim (E : f'; \bar{E} : g) \Rightarrow (E : f; \bar{E} : h) \succsim (E : f'; \bar{E} : h)$ hold for all $\sigma \cap \sigma'$ -measurable acts f, g, h and all $E \in \sigma \cup \sigma'$, then \succsim has an expected utility representations.*

Of course, we know that each of these representations collapses to an expected utility representation if the sure thing principle holds, where the sure thing principle requires $(E : f; \bar{E} : g) \succsim (E : f'; \bar{E} : g) \Rightarrow (E : f; \bar{E} : h) \succsim (E : f'; \bar{E} : h)$ hold for all $\sigma \cap \sigma'$ -measurable acts f, g, h and all $E \in \sigma \cap \sigma'$. So the weakening of the antecedent here consists in requiring the implication for a much smaller set of events, namely $\sigma \cup \sigma'$ instead of the larger set $\sigma \cap \sigma'$.

furthermore assumed that the notion of stochastic independence the agent subscribes to is that of weak independence as defined in Section 3.

Theorem 4 *There exist maxmin expected utility representations of preferences that satisfy ISR such that all σ/σ' -BdGM-mechanisms satisfy isolation, with σ being ambiguous. If some preferences that satisfy ISR are representable by the smooth model of ambiguity aversion, then σ cannot be ambiguous if all σ/σ' -BdGM-mechanisms satisfy isolation.*

Theorem 4 entails that the focus on the smaller set of mechanisms guides us out of the dilemma posed by the asymmetry of strong independence if we focus on preferences that are representable by a maxmin expected utility. Things look differently when we try to use them to elicit preferences that can be represented by the smooth model of ambiguity aversion. In this case the assumptions of ambiguity aversion and mutual stochastic independence exclude each other, also when using the weaker notion of independence.

Epstein [10] and Klibanoff et. al. [28] find that stochastic independence is compatible with maxmin expected utilities but not with the smooth model of uncertainty aversion. Here I find both models allow for the case of some algebra σ that is *even* strongly independent of another algebra σ' . But, while strong independence is the relevant notion of stochastic independence for RI-mechanisms it lacks some attractive features of stochastic independence “as we know it”. First and foremost consider symmetry. So while it is not feasible to speak of strongly independent ambiguous algebras in either of the two models under consideration things turn out to be less restrictive when considering the weak independence. Finally the disparity between the smooth model of ambiguity averse preferences and the maxmin expected utility model becomes more apparent. It is possible for two ambiguous algebras to be weakly independent in the maxmin expected utility model. The picture looks a lot bleaker for the case of the smooth model: in that case the algebra σ is weakly independent of the algebra σ' if and only if it is strongly independent. Consequently in the smooth model of ambiguity averse preferences, there is not even scope for two weakly independent algebras such that at least one of them is ambiguous. In a sense this difference between the two models of preference representation can be viewed as a vindication of the debate between Epstein [10] and Klibanoff et. al. [28].

7 RI-mechanisms and the Independence Axiom: Conclusion

There is already a large literature on the validity of the isolation hypothesis. As a first instance let me name the literature on “preference reversals” which was initiated by psychologists and interpreted by economists as evidence that isolation might not generally hold for Becker de Groot Marshak mechanisms. The phenomenon of “preferences reversals” was introduced by Lichtenstein and Slovic [29]: it consists of a pair of bets, say A and B with the feature that a large group of subjects prefers A to B in the direct comparison, while at the same time assigning a higher price to B than to A . Lichtenstein and Slovic [29] as well as Grether and Plott [18], who repeated some of Lichtenstein and Slovic’s experiments for an audience of economists, argue that this can only be viewed as evidence of intransitivities, given that subjects should like more money than less. Karni and Safra [23] and Holt [22] hold against Grether and Plott [18], that the root of the apparent intransitivity might lie in their method of preference elicitation: the BdGM-mechanism. Consider a subject who prefers bet B to bet A , picks $\$x$ when offered the choice between B and $\$x$ in the BdGM-mechanism, and picks A when offered the choice between $\$x$ and A in the BdGM-mechanism. *If* isolation holds the latter two choices can be interpreted as the statement of the preferences $A \succsim \$x$ and $\$x \succsim B$, which together with $B \succsim A$ yields a cycle.

But Karni and Safra [23] and Holt [22] show that isolation holds if and only if the subjects preferences satisfy the independence axiom. Karni and Safra [23] construct a rank dependent utility function that can explain the experimental outcome of Grether and Plott [18] without any reference to intransitivities.¹⁴

It should consequently come as some surprise that the even larger class of RI-mechanisms can satisfy isolation when the agents’ preferences systematically fail independence, as is

¹⁴This theoretical argument has been followed up by some empirical studies of isolation: Starmer and Sugden [39] construct a RI mechanism in which subjects are paid according to one of two binary choice problems with probability one half each. To allow for any conclusion about the isolation hypothesis two control groups faced the two choice problems as single choice experiments. The choice problems were designed such that an agent who considers the overall problem of the two choices *and* gives the typical Allais-paradox answers, a rank dependent utility maximizer for example, should not satisfy isolation. Interestingly they find that agents do satisfy isolation.

the case with ambiguity aversion. The reason why RI-mechanisms are not doomed in the present context is that the named results refer to the independence axiom for preferences over (objective) lotteries. Conversely, ambiguity aversion is usually conceptualized as a failure of the independence axiom on (ambiguous) acts. Indeed, Karni and Safra [23] and Holt [22] are silent on the matter whether the isolation hypothesis could hold for BdGM mechanism even if the subjects' preferences violate the independence axiom for (ambiguous) acts.

I did show in the present paper that it is theoretically sound to use RI-mechanisms to elicit ambiguity averse preferences whether represented via maxmin expected utilities or by the smooth model of ambiguity aversion. I argued that matters do become dicey when one demands that stochastic independence be a relation of symmetry. If one imposes this - apparently commonsensical - requirement on the agents preferences, then no RI-mechanism satisfies isolation with an ambiguity averse agents who follows the smooth model. Conversely, for the case of maxmin expected utility maximizers at least BdGM-mechanisms can be salvaged. In that case one can define mutually stochastically independent randomization devices such that any BdGM mechanism that uses such a device satisfies isolation.

8 Appendix

Proof of Lemma 1

(i) \Rightarrow (ii): Let σ and σ' be stochastically independent. Consider a σ/σ' -RI-mechanism. Given that the subject is an expected utility maximizer the utility for any choice $f = (f_1, \dots, f_n) \in \mathcal{S}$ can be expressed as

$$\begin{aligned}
 U((E_1 : f_1; \dots; E_n : f_n)) &= \sum_{i=1}^n \pi(E_i) \int_{\Omega} u(f_i(\omega)) d\pi(\omega | E_i) = \\
 &= \sum_{i=1}^n \pi(E_i) \int_{\Omega} u(f_i(\omega)) d\pi(\omega) = \sum_{i=1}^n \pi(E_i) u(x_{f_i}).
 \end{aligned}$$

The first equality follows by the definition of the expected utility representation U , the second and crucial equality follows from the assumption of stochastic independence, the third equality follows from the definition of the certainty equivalent x_{f_i} of an act f_i . Now

observe that $\sum \pi(E_i)u(x_{f_i})$ is maximized if each term of the sum is maximized, which holds, in turn, if the subject picks the most preferred act f_i^* out of each S_i . Consequently isolation holds for any σ/σ' -RI-mechanism.

(ii) \Rightarrow (iii): BdGM-mechanisms are special cases of RI-mechanisms.

(iii) \Rightarrow (i): Suppose that σ and σ' are not stochastically independent. Then there exist events $E \in \sigma'$ and $G \in \sigma$ such that $\pi(G | E) \neq \pi(G)$. Choose x, y such that $\pi(G)u(x) + \pi(\bar{G})u(y) < \pi(G | E)u(x) + \pi(\bar{G} | E)u(y)$ and define f such that $f(\omega) = x$ for $\omega \in G$ and $f(\omega) = y$ otherwise. Consider the BdGM-mechanism (f, g) with $g(\omega) = z$ for $\omega \in E$ and $g(\omega) = \max\{x, y\}$ otherwise, where z is chosen such that $U(f) = \pi(G)u(x) + \pi(\bar{G})u(y) < u(z) < \pi(G | E)u(x) + \pi(\bar{G} | E)u(y)$. The agent faces the choice among the following compound acts in the BdGM-mechanism: $g = (E : z, \bar{E} : \max\{x, y\})$, $(E : f, \bar{E} : \max\{x, y\})$, $(E : z, \bar{E} : f)$ and $f = (E : f, \bar{E} : f)$. If the BdGM-mechanism satisfied isolation the agent would have to chose g as $U(f) < u(z) < u(\max\{x, y\})$. However $U((E : f, \bar{E} : \max\{x, y\})) = \pi(E)(\pi(G | E)u(x) + \pi(\bar{G} | E)u(y)) + \pi(\bar{E})u(\max\{x, y\}) > \pi(E)u(z) + \pi(\bar{E})u(\max\{x, y\}) = U(g)$, so g cannot be chosen. \square

Proof of Theorem 1

(σ is strongly independent $\sigma' \Rightarrow$ all σ/σ' -RI mechanisms satisfy isolation): Suppose σ was strongly independent of σ' . Take some RI-mechanism $(\mathcal{E}, \mathcal{S})$, with $\mathcal{E} = (E_1, \dots, E_n)$ a σ' -measurable partition and $\mathcal{S} = (S_1 \times \dots \times S_n)$, sets of σ -measurable acts. Define f_1^* such that $f_1^* \succsim f_1$ for all $f_1 \in S_1$. Observe that

$$\begin{aligned} (E_1 : f_1^*; E_2 : f_n \cdots; E_n : f_n) &\sim (E_1 : x_{f_1^*}; E_2 : f_n \cdots; E_n : f_n) \succsim \\ (E_1 : x_{f_1}; E_2 : f_n \cdots; E_n : f_n) &\sim (E_1 : f_1; E_2 : f_n \cdots; E_n : f_n) \end{aligned}$$

holds for all $(f_1, \dots, f_n) \in (S_1 \times \dots \times S_n)$, with strong independence implying the two statements of indifference, and $f_1^* \succsim f_1$ together with (MON) implying the statement of weak preference. So the subject chooses from S_1 in the RI-mechanism as if he was facing the choice problem in isolation. The same arguments apply to the subjects choice in any of the other events; the RI-mechanism satisfies isolation.

(σ is strongly independent $\sigma' \Leftarrow$ all σ/σ' -RI mechanisms satisfy isolation): Suppose σ was not strongly independent of σ' . Then there must exist some σ/σ' -compound act $(E_1 :$

$f_1, E_2 : f_2, \dots, E_n : f_n$ such that either $(E_1 : f_1, E_2 : f_2, \dots, E_n : f_n) \succ (E_1 : x_{f_1}, E_2 : f_2, \dots, E_n : f_n)$ or $(E_1 : f_1, E_2 : f_2, \dots, E_n : f_n) \prec (E_1 : x_{f_1}, E_2 : f_2, \dots, E_n : f_n)$. I focus on the first case, the following arguments apply mutatis mutandis to the second case. (CON) allows me to define an x such that $x > x_{f_1}$ and $(E_1 : f_1, E_2 : f_2, \dots, E_n : f_n) \succ (E_1 : x, E_2 : f_2, \dots, E_n : f_n)$. (MON) implies that $(E_1 : x, E_2 : f_2, \dots, E_n : f_n) \succeq (E_1 : x_{f_1}, E_2 : f_2, \dots, E_n : f_n)$. Define a σ/σ' -RI-mechanism $(\mathcal{E}, \mathcal{S})$ such that $S_1 = \{x, f_1\}$ and $S_i = \{f_i\}$ for $i = 2, \dots, n$. Observe that taken together these assumptions imply that the agent chooses (f_1, f_2, \dots, f_n) when faced with the RI-mechanism. However the agent would have to choose x from S_1 in the single choice experiment as $x > x_{f_1} \sim f_1$, consequently $(\mathcal{E}, \mathcal{S})$ does not satisfy isolation.

(σ is weakly independent $\sigma' \Rightarrow$ all σ/σ' -BdGM mechanisms satisfy isolation): Fix a σ/σ' -BdGM mechanism (f, g) . Let the subject maximize his utility when naming x^* as the cutoff value to the experimenter. Formally, let $x^* \in X$ be defined such that

$$\left(\{\omega : g(\omega) \leq x^*\} : f; \{\omega : g(\omega) > x^*\} : g \right) \succeq \left(\{\omega : g(\omega) \leq x\} : f; \{\omega : g(\omega) > x\} : g \right).$$

for all $x \in X$. Since all events $\{\omega : g(\omega) \leq x^*\}$ are σ' -measurable, weak independence of σ from σ' implies that x^* also satisfies

$$\left(\{\omega : g(\omega) \leq x^*\} : x_f; \{\omega : g(\omega) > x^*\} : g \right) \succeq \left(\{\omega : g(\omega) \leq x\} : x_f; \{\omega : g(\omega) > x\} : g \right)$$

for all $x \in X$. (MON) implies that $x^* = x_f$ satisfies the latter line for all $x \in X$. So the the BdGM-mechanism (f, g) satisfies isolation. \square

The reason why weak independence is sufficient but might not be necessary for all BdGM-mechanisms to satisfy isolation is that the the assumption that all σ/σ' -BdGM-mechanisms satisfy isolation does not imply the indifference $(E : f, \bar{E} : g) \sim (E : x_f, \bar{E} : g)$ for all σ -measurable acts f , all σ' -measurable acts g and all events $E \in \sigma'$, but only for the cases in which $x_f \leq g(\omega)$ for all $\omega \in \bar{E}$. I next show that the even weaker notion of K-independence is necessary for all BdGM-mechanisms to satisfy isolation. This observation will be of use in some of the later proofs.

Definition 3 *An algebra σ is **K-independent** of another algebra σ' if*

$$(E : f; \bar{E} : x_f) \sim f \text{ holds for all } \sigma - \text{measurable acts } f \text{ and all } E \in \sigma'.$$

The letter K in this definition stands for Klibanoff [26] where this notion of stochastic independence was first proposed. Note that the definition of K-independence applies the indifference condition to a yet smaller set of cases than the definitions of weak and strong independence, namely only to a very particular subclass of the σ' -measurable acts g : the constant acts x_f . To see that σ needs to be K-independent of σ' for the set of all σ/σ' -BdGM-mechanisms to satisfy isolation suppose that σ was not even K-independent of σ' . Then there must exist an $E \in \sigma'$ and a σ -measurable act f , such that $(E : f; \bar{E} : x_f) \succ f$ or $(E : f; \bar{E} : x_f) \prec f$. Consider only the first of these two options, the arguments pertaining to the second follow analogously. (MON) and (CON) imply that there exist $y > x > x_f$ such that $(E : f; \bar{E} : x_f) \succ (E : x; \bar{E} : y)$. (MON) furthermore implies that $(E : f; \bar{E} : y) \succeq (E : f; \bar{E} : x_f)$ and $(E : x; \bar{E} : y) \succ x_f$. Define $g := (E : x; \bar{E} : y)$, and observe that the σ/σ' -BdGM (f, g) measures x_f to lie between x and y as $(\{\omega : g(\omega) \leq x\} : f; \{\omega : g(\omega) > x\} : g) = (E : f; \bar{E} : y) \succ (E : x; \bar{E} : y) = g \succ x_f$. Consequently the BdGM mechanism (f, g) does not satisfy isolation as $x, y > x_f$.

Proof of Theorem 2

(A \Leftrightarrow σ is strongly independent of σ'): Observe that under condition (A) the utility of any σ/σ' -compound act $(E_1 : f_1; E_2 : f_2; \dots E_n : f_n)$ can be calculated as follows:

$$\begin{aligned} & U((E_1 : f_1; E_2 : f_2; \dots E_n : f_n)) = \\ & \min_{\pi \in C} \int_{\Omega} u((E_1 : f_1; E_2 : f_2; \dots E_n : f_n)(\omega)) d\pi(\omega) = \\ & \min_{\pi_{\sigma'} \in C_{\sigma'}} \sum_{i=1}^n \pi_{\sigma'}(E_i) \min_{\pi_{\sigma}^i \in C_{\sigma}} \int_{\Omega} u(f_i(\omega)) d\pi_{\sigma}^i(\omega) = \\ & \min_{\pi_{\sigma'} \in C_{\sigma'}} \sum_{i=1}^n \pi_{\sigma'}(E_i) u(x_{f_i}) = U((E_1 : x_{f_1}; E_2 : x_{f_2}; \dots E_n : x_{f_n})). \end{aligned}$$

Consequently, (A) implies the strong independence of σ from σ' . Now consider violations of (A). First consider the case in which for some i , say $i = 1$, $\pi^i(\cdot | E_i)$ does not range over C_{σ} but some set D . Define f^* such that $\min_{\pi_{\sigma} \in C_{\sigma}} \int_{\Omega} u(f^*(\omega)) d\pi_{\sigma}(\omega) \neq \min_{\pi_{\sigma} \in D} \int_{\Omega} u(f^*(\omega)) d\pi_{\sigma}(\omega)$. In this case it is straightforward to define some act σ/σ' -compound act $(E_1 : f^*; E_2 : f_2; \dots E_n : f_n)$ such that $U((E_1 : f^*; E_2 : f_2; \dots E_n : f_n)) \neq U((E_1 : x_{f^*}; E_2 : f_2; \dots E_n : f_n))$ violating the strong independence of σ from σ' . Given that $\pi^i(\cdot | E_i)$ must range over C_{σ} , $U((E_1 : f_1; E_2 : f_2; \dots E_n : f_n))$ can be rewritten

as $\min_{\pi_{\sigma'} \in B} \sum_{i=1}^n \pi_{\sigma'}(E_i)u(x_{f_i})$ for some set B . For $\min_{\pi_{\sigma'} \in B} \sum_{i=1}^n \pi_{\sigma'}(E_i)u(x_{f_i})$ to equal $U((E_1 : x_{f_1}; E_2 : x_{f_2}; \dots E_n : x_{f_n}))$ for any vector $(x_{f_1}, x_{f_2}, \dots x_{f_n})$ we must have that $B = C_{\sigma'}$. Consequently (A) has to hold.

(B \Rightarrow σ is strongly independent of σ'): Consider any σ/σ' -compound act $(E_1 : f_1; E_2 : f_2; \dots E_n : f_n)$ and observe that

$$\begin{aligned} V((E_1 : f_1; E_2 : f_2; \dots E_n : f_n)) = \\ \int_{\Delta(\Omega), \pi(E_1)=1} \phi\left(\int_{\Omega} u((E_1 : f_1; E_2 : f_2; \dots E_n : f_n)(\omega))d\pi(\omega)\right)d\mu(\pi) + \\ \int_{\Delta(\Omega), \pi(E_1)=0} \phi\left(\int_{\Omega} u((E_1 : f_1; E_2 : f_2; \dots E_n : f_n)(\omega))d\pi(\omega)\right)d\mu(\pi): \end{aligned}$$

Now observe that the first term of the sum can be rewritten as follows:

$$\begin{aligned} \int_{\Delta(\Omega), \pi(E_1)=1} \phi\left(\int_{\Omega} u((E_1 : f_1; E_2 : f_2; \dots E_n : f_n)(\omega))d\pi(\omega)\right)d\mu(\pi) = \\ \int_{\Delta(\Omega), \pi(E_1)=1} \phi\left(\int_{\Omega} u(f_1(\omega))d\pi(\omega)\right)d\mu(\pi) = \\ \mu(\{\pi : \pi(E_1) = 1\}) \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f_1(\omega))d\pi(\omega)\right)d\mu(\pi | \{\pi : \pi(E_1) = 1\}) = \\ \mu(\{\pi : \pi(E_1) = 1\}) \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f_1(\omega))d\pi(\omega)\right)d\mu(\pi) = \\ \mu(\{\pi : \pi(E_1) = 1\})\phi(u(x_{f_1})) = \int_{\Delta(\Omega), \pi(E_1)=1} \phi(u(x_{f_1}))d\mu(\pi). \end{aligned}$$

The first and second equality follow from the restriction to probability measures π with $\pi(E_1) = 1$ and the definition of the conditional probability $\mu(\cdot | \{\pi : \pi(E_1) = 1\})$. The crucial assumption that $\mu_{\bar{\sigma}} = \mu_{\bar{\sigma}}(\cdot | \{\pi : \pi(E) = 1\})$ holds for all $E \in \sigma'$ implies the third equality. The fourth equality uses the definition of the certainty equivalent of f_1 . Taken together we obtain that

$$\begin{aligned} V((E_1 : f_1; E_2 : f_2; \dots E_n : f_n)) = \\ \int_{\Delta(\Omega), \pi(E_1)=1} \phi(u(x_{f_1}))d\mu(\pi) + \int_{\Delta(\Omega), \pi(E_1)=0} \phi\left(\int_{\Omega} u((f_1 : E_1 \dots f_n : E_n)(\omega))d\pi(\omega)\right)d\mu(\pi) = \\ \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u((E_1 : x_{f_1}; E_2 : f_2; \dots E_n : f_n)(\omega))d\pi(\omega)\right)d\mu(\pi). \end{aligned}$$

Since the same arguments apply to and $1 \leq i \leq n$ we can conclude that $V((E_1 : f_1; E_2 : f_2; \dots E_n : f_n)) = V((E_1 : x_{f_1}; E_2 : x_{f_2}; \dots E_n : x_{f_n}))$, and σ is strongly independent of σ' .

In the next part I show that considerably stronger claim that σ being K-independent of σ' implies the condition named in (B), the proof of this stronger claim is no more involved than the proof of the weaker claim but has the advantage to the proofs of Theorems 3 and 4 considerably easier.

(B \Leftarrow σ is K-independent of σ'): Normalize u and ϕ such that $u(0) = \phi(0) = 0$. Consider an ambiguous event $G \in \sigma$ and define a function $b(a)$ through $u(f(\omega)) = a$ for $\omega \in G$ and $f(u(\omega)) = b(a)$ for $\omega \notin G$ and $V(f) = 0$. (MON) and (CON) imply, that the function is well-defined, continuous, and strictly decreasing with $b(0) = 0$. Observe that the K-independence of σ from σ' implies that $f_E x_f \sim f$ and $x_{f_E} f \sim f$ holds for all such f and all $E \in \sigma$, implying

$$\begin{aligned}
0 &= U(f_E x_f) + U(x_{f_E} f) \\
\int_{\Delta(\Omega)} \phi\left(a\pi(G \cap E) + b(a)\pi(\overline{G} \cap E)\right) d\mu(\pi) + \int_{\Delta(\Omega)} \phi\left(a\pi(G \cap \overline{E}) + b(a)\pi(\overline{G} \cap \overline{E})\right) d\mu(\pi) &= \\
\int_{\Delta(\Omega)} \phi\left(a\pi(G \cap E) + b(a)\pi(\overline{G} \cap E)\right) + \phi\left(a\pi(G \cap \overline{E}) + b(a)\pi(\overline{G} \cap \overline{E})\right) d\mu(\pi) &\geq \\
\int_{\Delta(\Omega)} \phi\left(a\pi(G \cap E) + b(a)\pi(\overline{G} \cap E) + a\pi(G \cap \overline{E}) + b(a)\pi(\overline{G} \cap \overline{E})\right) d\mu(\pi) &= \\
\int_{\Delta(\Omega)} \phi\left(a\pi(G) + b(a)\pi(\overline{G})\right) d\mu(\pi) &= 0
\end{aligned}$$

The concavity of ϕ implies the weak inequality, a contradiction is achieved if the inequality holds strictly for some a . The strict concavity of ϕ together with the assumption that $\phi(0) = 0$ implies that $\phi(x) + \phi(y) > \phi(x + y)$ holds, if and only if $x \neq 0 \neq y$. Therefore the above inequality holds strictly if and only if $a\pi(G \cap E) + b(a)\pi(\overline{G} \cap E) \neq 0$ and $a\pi(G \cap \overline{E}) + b(a)\pi(\overline{G} \cap \overline{E}) \neq 0$ holds for a set of priors that has positive measure according to μ .

To avoid such a contradiction all $\pi \in \text{supp}(\mu)$ would have to belong to the following exhaustive list of cases in which either $a\pi(G \cap E) + b(a)\pi(\overline{G} \cap E) = 0$ or $a\pi(G \cap \overline{E}) + b(a)\pi(\overline{G} \cap \overline{E}) = 0$ holds for $a \neq 0$: 1. $\pi(E) = 0$, 2. $\pi(E) = 1$, or 3. $0 < \pi(E) < 1$ together with either $b(a) = -a \frac{\pi(G \cap E)}{\pi(\overline{G} \cap E)}$ or $b(a) = -a \frac{\pi(G \cap \overline{E})}{\pi(\overline{G} \cap \overline{E})}$, or both. To show that one of the first two conditions must hold for all $\pi \in \text{supp}(\mu)$ let us suppose to the contrary that there exists some $\pi^* \in \text{supp}(\mu)$ such that $0 < \pi^*(E) < 1$ and $b(a) \in \left\{ -a \frac{\pi^*(G \cap E)}{\pi^*(\overline{G} \cap E)}, -a \frac{\pi^*(G \cap \overline{E})}{\pi^*(\overline{G} \cap \overline{E})} \right\}$. This, together with the continuity of b together with $b(0) = 0$ imply that $b(a)$ is a piecewise linear function with $b(a) = -\rho a$ for all $a \geq 0$ for some fixed $\rho > 0$.

Now let $0 < a < 1$. Observe that G was chosen such that $\{\pi(G) : \pi \in \text{supp}(\mu)\}$ is not a singleton set (G is ambiguous). This implies in turn that $a\pi(G) + b(a)\pi(\overline{G}) \neq 0$ must hold for a positive measured subset of $\text{supp}(\mu)$. Consequently the strict concavity of ϕ around 0 implies the following contradiction:

$$0 = V((G : a, \overline{G} : b(a))) = \int_{\Delta(\Omega)} \phi(a\pi(G) + b(a)\pi(\overline{G}))d\mu(\pi) = \\ \int_{\Delta(\Omega)} \phi(a\pi(G) - \rho a\pi(\overline{G}))d\mu(\pi) < a \int_{\Delta(\Omega)} \phi(\pi(G) - \rho\pi(\overline{G}))d\mu(\pi) = 0.$$

We can conclude that $\pi(E)(1 - \pi(E)) = 0$ must hold for all $E \in \sigma'$ and $\pi \in \text{supp}(\mu)$.

Next suppose there was some event $E \in \sigma'$ such that $\mu_{\overline{\sigma}} \neq \mu_{\overline{\sigma}}(\cdot | E)$. This implies that there exists a σ -measurable act f such that $\int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega))d\pi(\omega)\right)d\mu(\pi) \neq \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega))d\pi(\omega)\right)d\mu(\pi | E)$. This in turn implies that for the given act f and the event E :

$$V((E : f; \overline{E} : x_f)) = \\ \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u((E : f; \overline{E} : x_f)(\omega))d\pi(\omega)\right)d\mu(\pi) = \\ \mu(\{\pi | \pi(E) = 1\}) \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega))d\pi(\omega)\right)d\mu(\pi | E) + \pi(\overline{E})\phi(u(x_f))d\mu(\pi) \neq \\ \mu(\{\pi | \pi(E) = 1\})\phi(u(x_f)) + \mu(\{\pi | \pi(E) = 0\})\phi(u(x_f)) = V(f),$$

a contradiction with the K-independence of σ from σ' . \square

Proof of Theorem 3

Let σ and σ' be mutually strongly independent. Let σ be ambiguous, that let there be at least one ambiguous event $G \in \sigma$. Let E be a non-null event in σ' . Since σ and σ' are mutually strongly independent also their subalgebras $\sigma_E := (\emptyset, E, \overline{E}, \Omega) \subset \sigma$ and $\sigma_G := (\emptyset, G, \overline{G}, \Omega) \subset \sigma'$ are mutually strongly independent.

Consider the case in which the agent's preferences are representable by a maxmin expected utility. For $(\emptyset, E, \overline{E}, \Omega)$ to be strongly independent of $(\emptyset, G, \overline{G}, \Omega)$ the set $C_{\sigma_E \cap \sigma_G}$ can be constructed as the convex hull of the set of priors $\pi^{(i,j,k)}$ for $i, j, k \in \{0, 1\}$ with the following definitions: $\pi^{(0,j,k)}(E) = \min_{\pi \in C} \pi(E)$ and $\pi^{(1,j,k)}(E) = \max_{\pi \in C} \pi(E)$ for all $j, k \in \{0, 1\}$; $\pi^{(i,0,k)}(E \cap G) = \pi^{(i,0,k)}(E) \times \min_{\pi \in C} \pi(G)$ and $\pi^{(i,1,k)}(E \cap G) = \pi^{(i,1,k)}(E) \times$

$\max_{\pi \in C} \pi(G)$ for all $i, k \in \{0, 1\}$; and $\pi^{(i,j,0)}(\overline{E} \cap G) = \pi^{(i,j,0)}(\overline{E}) \times \min_{\pi \in C} \pi(G)$ and $\pi^{(i,j,1)}(\overline{E} \cap G) = \pi^{(i,j,1)}(\overline{E}) \times \max_{\pi \in C} \pi(G)$ for all $i, j \in \{0, 1\}$. Therefore

$$\begin{aligned} \min_{\pi \in C} \pi(E | G) &= \min_{\pi \in C} \frac{\pi(E \cap G)}{\pi(G)} \leq \\ &\frac{\min_{\pi \in C} \pi(E) \min_{\pi \in C} \pi(G)}{\min_{\pi \in C} \pi(E) \min_{\pi \in C} \pi(G) + (1 - \min_{\pi \in C} \pi(E)) \max_{\pi \in C} \pi(G)} < \\ &\min_{\pi \in C} \pi(E) \end{aligned}$$

where the strict inequality holds since G is an ambiguous event so $\min_{\pi \in C} \pi(G) \neq \max_{\pi \in C} \pi(G)$.

Now define an act f through $u(f(\omega)) = a$ for $\omega \in E$ and $u(f(\omega)) = b$ otherwise, where $x_f = 0$ and $a > 0$. Let π^* be such that $\pi^*(E | G) = \min_{\pi \in C} \pi(E | G)$ and observe that $U((G : f, \overline{G} : 0)) \leq \pi^*(G)(a\pi^*(E | G) + b\pi^*(\overline{E} | G)) < \pi^*(G) \min_{\pi \in C} (a\pi(E) + b\pi(\overline{E})) = 0$. Therefore σ' cannot even be K-independent of σ when the ambiguous algebra σ is strongly independent of σ' .

Now consider the case of preferences that are representable by the smooth model together with a compound act $(G : f; \overline{G} : 0)$ where 0 stands for the constant act that yields 0 utility and f is a bet on E such that $f(\omega) = x$ for $\omega \in E$ and $f(\omega) = y$ otherwise, and $V(f) = 0$. Since σ' is strongly independent of σ we obtain the following by Theorem 2:

$$\begin{aligned} V(f) &= \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega)) d\pi(\omega)\right) d\mu(\pi) = \\ &\mu(\{\pi : \pi(E) = 1\})\phi(u(x)) + \mu(\{\pi : \pi(E) = 0\})\phi(u(f(y))) = 0 \end{aligned}$$

and

$$\begin{aligned} V((G : f; \overline{G} : 0)) &= \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u((G : f; \overline{G} : 0)(\omega)) d\pi(\omega)\right) d\mu(\pi) = \\ &\mu(\{\pi : \pi(E) = 1\}) \int_{\Delta(\Omega)} \phi(\pi(G)u(x)) d\mu(\pi) + \mu(\{\pi : \pi(E) = 0\}) \int_{\Delta(\Omega)} \phi(\pi(G)u(y)) d\mu(\pi) \end{aligned}$$

The strong independence of σ from σ' implies that $V((G : f; \overline{G} : 0)) = V((G : x_f; \overline{G} : 0)) = V((G : 0; \overline{G} : 0)) = 0$. In sum we obtain that

$$\begin{aligned} \mu(\{\pi : \pi(E) = 1\})\phi(u(x)) + \mu(\{\pi : \pi(E) = 0\})\phi(u(f(y))) = 0 = \\ \mu(\{\pi : \pi(E) = 1\}) \int_{\Delta(\Omega)} \phi(\pi(G)u(x)) d\mu(\pi) + \\ \mu(\{\pi : \pi(E) = 0\}) \int_{\Delta(\Omega)} \phi(\pi(G)u(y)) d\mu(\pi) \end{aligned}$$

which can only hold for all bets f on E with $V(f) = 0$ if \succsim has an expected utility representation with respect to G , with $\pi(G) \in \{0, 1\}$ for all $\pi \in \text{supp}(\mu)$. \square

Proof of Theorem 4

To see the first claim observe that σ and σ' are weakly mutually independent for any maxmin expected utility representation with a set of beliefs $C = C_\sigma \times C_{\sigma'}$.

To see the second claim observe that the proof of Theorem 2 implies that σ is strongly independent of σ' if and only if σ is very weakly independent of σ' . Theorem 3 established that there cannot be any mutually strongly independent algebras when at least one of them is ambiguous and when the agent's preferences are representable by the smooth model of ambiguity aversion. Taken together these two observations imply the existence of two mutually very weakly independent algebras with at least one of them being ambiguous is incompatible with the smooth model of ambiguity aversion. This yields the result as we know from the remarks after the proof of Theorem 1 that very weak independence is a necessary condition for BdGM-mechanisms to satisfy isolation. \square

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