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Safe Assets, Risky Assets, and Dynamic Inefficiency in Overlapping-Generations Economies*

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Abstract

The paper gives conditions for dynamic inefficiency of *laissez-faire* allocations in an overlapping-generations model with safe and risky assets. If the rate of population growth is certain, the conditions given depend *only* on how the rate of return on safe assets compares to the growth rate. If no safe assets are held, the implicit relative price for non-contingent intertemporal exchanges takes the place of the safe rate of return. Returns on risky assets do not enter the comparison. The conclusion holds regardless of whether welfare assessments are made from an *interim* perspective, taking account of the information that people have, or from an *ex ante* perspective. If a *laissez-faire* allocation is dynamically inefficient, a Pareto improvement can be implemented by a suitable fiscal policy intervention, which includes specific taxes or subsidies that neutralize incentive effects on risky investments and the price effects they induce.

Key Words: Dynamic Inefficiency, overlapping-generations models, safe asset shortages, macro risk allocation, public debt.

JEL: D15, D61, E21, E22, E62, H30.

1 Introduction

In overlapping-generations models with infinite time horizons, equilibrium allocations under *laissez-faire* need not be Pareto efficient.[1] Such "dynamic inefficiency" is often tied to the question whether the real rate of return r on

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[1] The argument goes back to Allais (1947, Appendix 2), Samuelson (1958), and Diamond (1965). Blanchard (2019) as well as von Weizsäcker (2014) and Weizsäcker and Krämer (2019) have provided the discussion with a new impetus.

capital is smaller or larger than the real growth rate g of the economy. If $r < g$, efficiency can be improved by reducing capital investments in all periods and using the resources that are thereby saved to provide for the consumption of older participants. The "rate of return" that any agent achieves by participating in this package, reducing capital investment in one period in order to provide for older participants' consumption and receiving a "return" in the form of payments from younger people in the future, who in turn reduce their capital investments, is equal to the growth rate g, which exceeds the rate of return on capital investments if $r < g$.

The argument is unproblematic if all real assets are riskless so that in equilibrium they all bear the same rate of return. The argument is unclear, however, if some assets, or even all, are risky so that their rates of return are given by random variables, rather than real numbers. What are we to conclude if the equilibrium rate of return on safe assets is smaller than the growth rate of the economy and the expected rates of return on risky assets are larger than the growth rate of the economy? The question is particularly important today because developments of the past few decades have created a situation where by all accounts real rates of return on safe assets are below real growth rates while mean rates of return on portfolios of risky assets have remained above real growth rates.

Following Abel et al. (1989), most of the literature presumes that assessments of dynamic inefficiency must consider returns on all assets, risky as well as riskless.[2] This paper, however, will show that only the riskless rate of return is relevant. If no riskless asset is held, the intertemporal marginal rate of substitution between consumption levels in successive periods takes the place of the riskless rate of return.

Contrary claims in the literature are based on misunderstandings. Misunderstandings arise because it is easy to mix up Pareto efficiency with production efficiency in the sense of Koopmans (1953) and Cass (1972), which only deals with the question whether consumption at some date can be increased without lowering consumption at some other date. The introductory paragraph concerns a failure of production efficiency. A failure of production efficiency implies a failure of Pareto efficiency, but the converse is not true. A finding of production efficiency is not sufficient for Pareto efficiency.

Abel et al. (1989) gave an example to show that the safe rate of return is irrelevant. The example involves an infinitely-lived representative consumer who owns a Lucas tree. The tree yields a random crop of non-storable fruit in each period, which the consumer eats. The allocation is fixed but may be deemed to be the outcome of a no-trade equilibrium in a complete system of Arrow-Debreu markets. The associated Arrow-Debreu prices can be used to compute a sequence of riskless interest rates. In the example, these interest rates take a constant value less than one, and yet the equilibrium allocation is efficient. Production efficency holds trivially because there are no production

[2] In addition to Abel et al. (1989), see, e.g., Homburg (2014), Geerolf (2018), Blanchard (2019), Yared (2019), Acharya and Droga (2020), Reis (2020).

choices to be taken. Pareto efficiency holds trivially because there is only one agent.

However, dynamic inefficiency involves multiple consumers, in fact, a *large-square economy*,[3] where the infinity of goods (at least one per period) is matched by an infinity of consumers (at least two for every good). To understand why, consider the "infinite hotel" that the mathematician Hilbert introduced to explain the notion of infinity. According to Hilbert, an infinite hotel is one in which one can always accommodate another guest, even when the hotel is already full. Put the new guest into room 1, the guest from room 1 into room 2, and so on...

Modifying Hilbert's idea, one can also define an infinite hotel as one that is fully occupied by different guests and yet the allocation of rooms need not be Pareto efficient. For suppose that the guests have preferences over rooms, such that, for each n, the guest in room n prefers room $n+1$ over room n. Then a Pareto improvement can be obtained by moving the guest from room 1 into room 2, the guest from room 2 into room 3, and so on.

This argument uses the infinite number of participants as well as the infinite number of rooms. If the hotel was occupied by a king with an infinite retinue, and only the preferences of the king mattered, there would be no scope for Pareto inefficiency. There is also no scope for Pareto inefficiency if rooms 1 to 100 are occupied by independent individuals and the king with his infinite retinue occupies all the other rooms. In this case, the attempt to move guest 100 into room 101 would make the king worse off because there is nobody to compensate him for the loss of room 100.

The infinitely-lived representative consumer in the example of Abel et al. (1989) is like the king in Hilbert's hotel. Therefore, the example merely illustrates the fact that inefficiency of competitive-equilibrium allocation requires not only many goods (rooms), but also many participants so that there is always another "spare" person to compensate the last person imposed upon.[4]

The infinite hotel with an infinite number of independent guests is similar to the overlapping-generations economy in Samuelson's (1958) consumption loan model without capital. In the simplest version of this model, there is a single perishable good in each period. People live for two periods and have an endowment of the good in the period in which they are born and no endowment in the next period. The autarky allocation, in which every participant consumes the endowment in the first period of life and consumes nothing in the second period of life, is an equilibrium allocation under *laissez-faire* but is Pareto-dominated by an allocation in which every participant gives some of the initial endowment to older people and in the next period receives some of the initial endowment

[3] See Balasko and Shell (1980).

[4] In addition to, and separately from, the example intended to show that the safe rate does not matter, Abel et al. (1989) also have a sufficient condition for Pareto efficiency in overlapping-generations models. In Section 4 below, I show that, in the model of this paper, their condition is a special case of my condition on the safe rate of return or the intertemporal marginal rate of substitution attached to non-contingent changes in consumption in different periods.

of the next generation. Moving from the autarky allocation to one where people of each generation share some of their endowments with older people is the analogue of person n moving from room n to room $n+1$ in Hilbert's hotel.

Like the infinite hotel, this version of the consumption loan model does not involve capital, so the inefficiency of *laissez-faire* allocations cannot be due to overaccumulation. The inefficiency concerns the allocation of different consumption goods (rooms) over different people. Whereas the introductory argument involved a failure of production efficiency, the consumption loan model indicates that, in a large-square economy, there can be a failure of Pareto efficiency even if there is no production and even if production efficiency holds.[5]

Ultimately, dynamic inefficiency is due to the fact that equilibrium prices only provide local signals about relative scarcities and, in a large-square economy, these local signals can provide misleading information about overall scarcities. Implementation of a Pareto efficient allocation may require coordination among *all* participants, and, in the absence of further devices, this coordination need not be available under *laissez faire*.[6]

To study dynamic inefficiency in the presence of uncertainty, I use a model with overlapping generations in which people live for two periods. In the first period, they supply labour inelastically and use their wage income for consumption and for investment, in the second period, they consume the returns from their investments. There are two real assets, safe and risky. Risks are due to productivity shocks. These shocks represent aggregate risk, so there is no question of diversification and risk sharing between participants.

The productivity shocks affect the capital owners' labour demands and therefore the market-clearing wage rates, and the incomes available to the members of the next generation. Equilibrium is defined in terms of a stochastic process of wage rates, as well as consumption, investment and labour choices of the members of different generations in different periods.

The assumptions I impose on preferences and technologies ensure the existence of a unique equilibrium. The equilibrium reflects the stationary recursive structure of the model. Specifically, the equilibrium wage process is a time-homogeneous Markov process with a unique invariant distribution to which the probability distributions of wage rates in periods $t = 1, 2, ...$ converge. Events in any one period are fully determined by the productivity shock and wage rate in that period and by people's expectations of the productivity shock and wage rate in the next period.

[5] For a clear exposition of the distinction, see Chapters 20.C and 20.H in Mas-Colell et al. (1995).

[6] This is not just a matter of people all meeting together. Even if all participants were able to meet and trade in a single system of Walrasian markets, equilibrium allocations need not be Pareto efficient. For example, in Samuelson's consumption loan model, the autarky allocation is a Walrasian equilibrium even though it is Pareto dominated. The associated equilibrium price system is such that, even though each agent's budget constraint is well specified, the equilibrium value of the aggregate goods endowment is unbounded. The standard argument in the proof of the First Welfare Theorem, that something that is better for all must also cost more on aggregate, is thus invalid. See, e.g., Balasko and Shell (1980), Mas-Colell et al. (1995), Ch. 20.H.

In this setting, the question whether the *laissez-faire* equilibrium allocation is Pareto efficient can be asked in several different ways, depending on the information on which the assessment is conditioned and on the kind of change in allocation that is considered admissible. It makes a difference whether people born in period t are taken to assess a presumed Pareto improvement from an *interim* perspective, where they know the realizations of productivity shocks and wage rates up to and including period t, or from an *ex ante* perspective, without this information. It also makes a difference whether or not a change of allocation in period t is allowed to condition on productivity shocks and wage rates up to and including period t. If such conditioning is allowed and the welfare assessment is based on an *ex ante* perspective, one trivially finds that *laissez-faire* equilibrium allocations are generically Pareto inefficient because there is no mechanism for efficient sharing of the risks from period $t + 1$ productivity shocks between generations t and $t + 1$.

To avoid this trivial conclusion, I use the concept of *constrained Pareto efficiency* of Diamond (1967) and Hart (1975) and ask only whether Pareto improvements can be obtained by *non-contingent changes* in allocations. I ask this question from both an *interim perspective* and an *ex ante* perspective. From *both* perspectives, the answers depend on comparisons of intertemporal marginal rates of substitution attached to non-contingent changes in consumption in the first period and the second period of life to the population growth rate, which is assumed to be zero. If the riskless asset is held with probability one in each period, these intertemporal marginal rates of substitution are always equal to the rate of return on the riskless asset, so the question is how this rate of return compares to the population growth rate. If in some period and event the riskless asset is not held, the relevant intertemporal marginal rate of substitution is a function of the current state of the world.

From an *interim* perspective, dynamic inefficiency arises if the intertemporal marginal rate of substitution is less than one in *all* states of the world. From an *ex ante* perspective, dynamic inefficiency arises if a marginal-utility-weighted expected value of the intertemporal marginal rate of substitution is less than one, where expectations are taken with respect to the invariant distribution of the equilibrium wage process. None of these comparisons has anything to do with returns on risky assets.

Moving on from the question whether a given equilibrium allocation is Pareto efficient to the question whether a Pareto improvement can be implemented by a fiscal intervention, I consider a package consisting of lump sum taxes on people when they are young, lump sum subsidies to people when they are old, and specific subsidies to or taxes on risky investments, which are specified so that they exactly neutralize the incentive effects of the lump sum taxes and transfers on risky investments. This neutralization is always possible; moreover, it can be done without violating the government budget constraint.

Such a combination of lump sum taxes, lump sum subsidies, and specific subsidies or taxes is shown to induce a Pareto improvement if and only if the *laissez-faire* equilibrium allocation fails to be dynamically efficient, i.e., if and only if the intertemporal marginal rate of substitution attached to a pair of non-contingent changes in consumption in the two periods of a person's life is smaller than the growth rate of the economy - in all states of the world if the *interim* approach is taken, in marginal-utility-weighted expectations under the invariant distribution of the equilibrium wage process if the *ex ante* approach is taken.

The analysis bears some similarity to that of Blanchard (2019). However, there are two important differences (besides the fact that I fully spell out the stochastic properties of equilibria). First, Blanchard does not distinguish between the question whether a given equilibrium allocation is Pareto efficient and the question whether a Pareto improvement can be implemented by a certain fiscal intervention. Second, he only considers a fiscal intervention with a lump sum tax on people when they are young and a lump sum subsidy to people when they are old, without the specific subsidies or taxes that I use to neutralize certain incentive effects.

For such an intervention, Blanchard distinguishes between *direct effects* and *price effects*, i.e., effects that come about because changes in behaviour induce changes in market-clearing prices. For example, the combination of a lump sum tax on people when they are young and a lump sum subsidy to people when they are old may induce the members of each generation to reduce their capital investments. Such a reduction in turn would reduce wage rates in the next period and therefore the wage incomes of the next generation.

For the direct effects of the intervention he considers, Blanchard's analysis leads to the same conclusions as mine, i.e., the assessment of constrained Pareto efficiency depends on the same comparison of non-contingent intertemporal marginal rates of substitution with growth rates. For the price effects, however, he reaches a different conclusion, with an assessment that depends on the rate of return on risky assets. The tradeoff between the two effects is ambiguous.

I avoid this ambiguity by focusing on fiscal interventions that leave risky investments unchanged so that there are no price effects. I do so because there is little one can say about price effects. Even their sign is unclear. Blanchard (2019) suggests that there is always a reduction of risky investments, which subsequently depresses wage rates, but this conclusion is due to his considering only one form of capital, which is safe in one specification and risky in another. Allowing for people to invest in safe and risky investments at the same time, I find that, if the safe rate of return is below the growth rate, a lump sum tax on people when they are young and a lump sum subsidy to people when they are old will induce *crowding in* of risky investments if safe investments are positive under *laissez faire* and crowding out of risky investments if safe investments are zero under *laissez faire.* In the first case, the subsequent wage rate is raised, in the second case lowered by the fiscal intervention.

Even if the sign of the price effects of the fiscal intervention is clear, e.g., if we know that risky investments are crowded out, no clear welfare assessment can be given. Blanchard's assessment suffers from the fact that he looks at specific realizations of the overall stochastic process for consumption, investment and rates of return with the property that, along those realizations, outcomes are constant over time. His criterion of whether the rate of return on risky investments exceeds the growth rate or not refers to the realized rate of return, rather than any measure of the return random variables that would drive investment decisions.

The price effects of the fiscal intervention involve not merely the distribution of income but also the distribution of risks between capital owners and wage earners. Risks from the random variable governing productivity in period t must be shared between the members of generation $t-1$, who own the capital used in period t, and the members of generation t, who are employed to work with this capital. A reduction of risky investments in period $t-1$ not only raises the expected rate of return on these investments (and lowers the expected wage rate in period t), but also raises the variance of the rate of return (and lowers the variance of the wage rate in period t).

Any welfare assessment must therefore take account of whether the change in the distribution of risks that is associated with price effects will or will not improve the risk allocation. This cannot be done from an *interim* perspective where the value of the relevant productivity random variable is already known. An assessment of price effects on the basis of this information would be comparable to an end-of-vacation assessment of accident insurance for the vacation, which obviously depends on whether the vacationer did or did not have an accident.

An assessment of changes in risk allocation from an *ex ante* perspective would be appropriate, but no clear assessment can be given. Because the market system is incomplete, the *laissez faire* allocation of risks is unlikely to be efficient. Moreover, there is no presumption as to whether the allocation is improved by shifting risks from wage earners to capital owners or from capital owners to wage earners. These difficulties however have more to do with the problems of risk allocation in an incomplete market system than with dynamic inefficiency.

In the following, Section 2 introduces the formal model. Section 3 establishes a few descriptive properties of *laissez-faire* equilibrium. Section 4 shows that the efficiency of *laissez-faire* equilibrium allocations depends on whether intertemporal marginal rates of substitution attached to non-contingent changes in consumption in the first period and the second period of life exceed or fall short of the population growth rate. For *ex ante* efficiency, the comparison concerns a marginal-utility weighted average of these intertemporal marginal rates of substitution. Section 5 discusses the implementation of Pareto improvement by fiscal interventions, showing that, in those cases where *laissez-faire* equilibrium allocations are not Pareto efficent, Pareto improvements can always be implemented by suitable combinations of lump sum taxes and transfers and of specific subsidies or taxes that neutralize the incentive effects of the intervention on risky investments.

Finally, Section 6 gives an informal discussion of several issues, the underlying reason for why the safe rate of return matters, the issue of crowding out versus crowding in of risky investments by fiscal interventions, the welfare assessment of price effects, the use of payroll-based rather than lump sum taxes and transfers, the use of public debt, the potential role of fiat money and, finally, the question whether the currently prevailing constellation of interest rates and growth rates can be taken as evidence of dynamic inefficiency.

All proofs are given in Appendix A. Appendix B gives a detailed account of *interim* and *ex ante* welfare assessments of price effects.

2 The Model

2.1 Basics

Consider an economy in periods $t = 1, 2, ...$ In each period t, there is a single produceable good, which serves for consumption and for investments. There are two kinds of investments, safe and risky. An investment k_s in the safe technology in period t generates an output rk_s in period $t+1$ where $r > 0$. An investment k_r in the risky technology in period t can be combined with a labour input ℓ_r in period $t+1$ to generate an output $A_{t+1}F(k_r, \ell_r)$ in period $t+1$, where A_{t+1} is the realization of a nondegenerate random variable $\tilde{A}_{t+1}$ with values in a compact interval $[\underline{A}, \bar{A}] \subset \mathbb{R}_+$.[7] After production, both capital goods are fully depreciated. The production function F is assumed to be twice continuously differentiable, increasing, and strictly quasi-concave, and to have constant returns to scale, with $F(k_r, 0) = F(0, \ell_r) = 0$ for all k_r and ℓ_r. The partial derivatives F_k, F_ℓ of F with respect to k_r and ℓ_r satisfy

$$\lim_{k_r \to 0} F_k(k_r, \ell_r) = \infty \quad \text{and} \quad \lim_{k_r \to \infty} F_k(k_r, \ell_r) = 0 \quad \text{for all } \ell_r > 0 \tag{2.1}$$

as well as

$$\lim_{\ell_r \to 0} F_\ell(k_r, \ell_r) = \infty \quad \text{and} \quad \lim_{\ell_r \to \infty} F_\ell(k_r, \ell_r) = 0 \quad \text{for all } k_r > 0. \tag{2.2}$$

I also assume that

$$k_r F_{kk}(k_r, \ell_r) + F_k(k_r, \ell_r) > 0 \tag{2.3}$$

for all k_r and ℓ_r so that, for any ℓ_r, the function

$$k_r \mapsto k_r F_k(k_r, \ell_r) \tag{2.4}$$

is strictly increasing.[8]

In each period t, a new generation of N_t people is born and lives for two periods. For simplicity, I assume that $N_t = N$ is a constant for all t, so the population growth rate is zero. However, I will carry the symbol N along as

[7] The asymmetry in the specification of safe and risky technologies serves to immunize the rate of return on investment in the safe technology from the labour market repercussions of shocks in the risky technology. Without this asymmetry, returns to the safe technology would be affected by shocks to wage rates that are due to productivity shocks in the risky technology, i.e., these returns would not be riskless after all.

[8] Equivalently, the elasticity of substitution is greater than the labour share of income under competitive input pricing. For CES production functions, the condition holds globally if and only if the elasticity of substitution is no smaller than one.

it makes it easier to see the distinction between market-clearing conditions and budget constraints. There are also $N_0 = N$ old people in period 1.

For simplicity, I assume that, apart from their birth dates, all people have the same characteristics. A person born in period t has initial endowments $e \geq 0$ of the produceable good and $\bar{L}$ of labour. In period t, the person offers an amount $\bar{L}$ of labour in the market; the person also chooses a current consumption level c_1^t, and investments k_s^t and k_r^t in the safe and risky technologies under the constraint

$$c_1^t + k_s^t + k_r^t = e + w_t \bar{L}, \tag{2.5}$$

where w_t is the prevailing wage rate. In period $t+1$, the person hires labour ℓ_r^t at the then prevailing wage rate w_{t+1} and uses this labour for production with the risky technology. The person's second-period consumption c_2^t is given by the excess of total output over the wage bill,

$$c_2^t = rk_s^t + A_{t+1}F(k_r^t, \ell_r^t) - w_{t+1}\ell_r^t. \tag{2.6}$$

Given the choice vector $(c_1^t, k_s^t, k_r^t, \ell_r^t, c_2^t)$, the person obtains the utility

$$u(c_1^t) + v(c_2^t). \tag{2.7}$$

The utility functions $u(\cdot)$ and $v(\cdot)$ are assumed to be twice continuously differentiable, increasing and strictly concave, with $u'(0) = \infty$ and $v'(0) = \infty$ and with second derivatives that are bounded away from zero. Moreover, $v(\cdot)$ exhibits non-increasing absolute risk aversion, i.e, the function

$$c \longmapsto -\frac{v''(c)}{v'(c)} \tag{2.8}$$

is nonincreasing.

An old person in period 1 has an initial endowment consisting of k_s^0 units of capital in the safe technology and k_r^0 units of capital in the risky technology. This person hires labour ℓ_r^0 at the wage rate w_1 so as to obtain the consumption

$$c_2^0 = rk_s^0 + A_1F(k_r^0, \ell_r^0) - w_1\ell_r^0, \tag{2.9}$$

in accord with (2.6).

I assume that the productivity parameters $\tilde{A}_1, \tilde{A}_2, ...$ are independent and identically distributed, with a common probability distribution P such that $\underline{A}$ is the smallest element and $\bar{A}$ is the largest element of the support of P. The common mean $\int AdP(A)$ of the random variables $\tilde{A}_1, \tilde{A}_2, ...$ will be denoted as A^*.

In the beginning of each period t, the value of $\tilde{A}_t$ becomes known, and choices in this period can be based on this information (as well as information about $\tilde{A}_1, ..., \tilde{A}_{t-1}$). These parameters are taken to be the same for all participants, so there is no issue of sharing idiosyncratic risk. The aggregate risk from the productivity parameter $\tilde{A}_t$ in period t is shared between generations $t-1$ and t according to the division of output under competitive pricing.

Because the productivity parameters $\tilde{A}_1, \tilde{A}_2, ...$ affect all production with risky capital, they will also affect equilibrium wage rates in all periods. The wage rates $w_1, w_2, ...$ must therefore be seen as the realizations of random variables $\tilde{w}_1, \tilde{w}_2, ...$ In view of the constraints (2.5) and (2.6), it follows that the choice vector $(c_1^t, k_s^t, k_r^t, \ell_r^t, c_2^t)$ must be seen as the realization of a random vector $(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)$, a contingent plan that indicates how the person acts on the information about productivity parameters and wage rates that he or she receives.

2.2 Optimization and Equilibrium

From the perspective of individuals, the random variables $\tilde{A}_1, \tilde{A}_2, ...$ and $\tilde{w}_1, \tilde{w}_2, ...$ are exogenous. An individual that is born in period t must choose a plan $(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)$ so as to maximize the expected utility

$$E[u(\tilde{c}_1^t) + v(\tilde{c}_2^t)] \tag{2.10}$$

subject to the constraints

$$\tilde{c}_1^t + \tilde{k}_s^t + \tilde{k}_r^t = e + \tilde{w}_t \bar{L}, \tag{2.11}$$

and

$$\tilde{c}_2^t = r\tilde{k}_s^t + \tilde{A}_{t+1} F(\tilde{k}_r^t, \tilde{\ell}_r^t) - \tilde{w}_{t+1} \tilde{\ell}_r^t, \tag{2.12}$$

which must all be satisfied almost surely for all t. The plan $(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)$ must also be adapted to the information the individual has when implementing a choice; thus $\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t$ can only depend on information available at t, and $\tilde{\ell}_r^t, \tilde{c}_2^t$ can only depend on information available at $t+1$.

Similarly, an old individual in period 1 must choose a pair $(\tilde{\ell}_r^0, \tilde{c}_2^0)$ so as to maximize the utility $v(\tilde{c}_2^0)$ under the constraint

$$\tilde{c}_2^0 = rk_s^0 + \tilde{A}_{t+1} F(k_r^0, \bar{L}) - \tilde{w}_1 \tilde{\ell}_r^0, \tag{2.13}$$

which must be satisfied almost surely.

A plan that solves the specified optimization problem for a given person will be called a *best response of that person to the sequence* $\{\tilde{A}_t, \tilde{w}_t\}_{t=1}^{\infty}$ of random productivity parameters and wage rates.

An *allocation* is an array of plans for all participants. An allocation is called *symmetric* if it stipulates the same plan for people of the generation. A symmetric allocation with plans $(\tilde{\ell}_r^0, \tilde{c}_2^0), (\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t), t = 1, 2, ...$ is *feasible*, if the actions $\tilde{\ell}_r^{t-1}, \tilde{c}_2^{t-1}, \tilde{L}^t, \tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t$ that are planned for period t only depend on information available at t and, moreover, for $t = 1, 2, ...,$

$$N \cdot \tilde{c}_2^{t-1} + N \cdot (\tilde{c}_1^t + \tilde{k}_s^t + \tilde{k}_r^t) = N \cdot (r\tilde{k}_s^{t-1} + \tilde{A}_t F(\tilde{k}_r^{t-1}, \tilde{\ell}_r^{t-1})) + N \cdot e \tag{2.14}$$

and

$$N \cdot \tilde{\ell}_r^{t-1} = N \cdot \bar{L}, \tag{2.15}$$

almost surely. Condition (2.14) ensures that resources devoted to consumption in any period are compatible with what is available from production and endowments. Condition (2.15) provides for the compatibility of labour demands and labour supplies.

An *equilibrium* is given by a sequence $\{\tilde{w}_t\}_{t=0}^{\infty}$ of state-contingent wage rates and a feasible symmetric allocation $(\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$, such that for any $t \geq 1$, the plan $(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)$ for a person born in period t is a best response of that person to the sequence $\{\tilde{A}_t, \tilde{w}_t\}_{t=1}^{\infty}$ of productivity parameters and wage rates and so is the plan $(\tilde{\ell}_r^0, \tilde{c}_2^0)$ for an old person in period 1.

To simplify the presentation, I will neglect the inessential multiplicity of best-response plans that arises because any two plans provide the same expected utility if they differ only on an event of probability zero. Readers who care about the difference are invited to add the terms "almost surely" and "essentially", as in "essentially unique", wherever appropriate.

3 Descriptive Properties of Equilibrium

3.1 Recursiveness

The model has a stationary recursive structure. In any period t, the old generation wants to hire an aggregate of $N \cdot \tilde{\ell}_r^{t-1}$ units of labour. The aggregate offer of the young generation is $N \cdot \tilde{L}^t$. The wage rate $\tilde{w}_t$ equilibrates the labour market. This wage rate determines the budget $e + \tilde{w}_t \bar{L}$ of a young person in period t. Apart from preferences and technology, this person's maximization problem depends only on the budget $e + \tilde{w}_t \bar{L}$ and on the person's expectations about the random pair $(\tilde{A}_{t+1}, \tilde{w}_{t+1})$ that will prevail in period $t+1$. The solution to this maximization problem will however determine the amount of risky capital that is held from period t to period $t+1$ is put to work in period $t+1$. This risky capital affects the wage rate $\tilde{w}_{t+1}$ in period $t+1$. The wage rate $\tilde{w}_{t+1}$ in period $t+1$ then determines the budget $e + \tilde{w}_{t+1} \bar{L}$ of a young person in period $t+1$. Apart from the fact that $\tilde{w}_{t+1}$ is typically different from $\tilde{w}_t$, that person's maximization problem has the same structure as the maximization problem of a young person in period t.

The following proposition shows that this recursive structure of the model is reflected in the structure of equilibrium.

Proposition 3.1 *Under the given assumptions, there exists a unique equilibrium. The equilibrium takes the form*

$$\tilde{w}_1 = \varphi(\tilde{A}_1, k_r^0) \tag{3.1}$$

and, for $t \geq 1$,

$$\tilde{w}_{t+1} = \psi(\tilde{A}_{t+1}, \tilde{w}_t), \tag{3.2}$$

$$\tilde{c}_1^t = c_1^*(\tilde{w}_t), \tilde{k}_s^t = k_s^*(\tilde{w}_t), \tilde{k}_r^t = k_r^*(\tilde{w}_t), \tag{3.3}$$

$$\tilde{\ell}_r^t = \lambda\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right) \cdot \tilde{k}_r^t, \tag{3.4}$$

and

$$\tilde{c}_2^t = r\tilde{k}_s^t + \tilde{A}_{t+1} \cdot \rho\left(\tilde{w}_t\right) \cdot \tilde{k}_r^t, \tag{3.5}$$

where the functions $\varphi(\cdot), \psi(\cdot), c_1^*(\cdot), k_s^*(\cdot), k_r^*(\cdot), \lambda(\cdot), \rho(\cdot)$ *satisfy:*

(i) For any $A \in [0, \bar{A}]$ *and any* $k_r > 0$,

$$\varphi(A, k_r) = A \cdot F_\ell(k_r, \bar{L}). \tag{3.6}$$

(ii) For any $A \in [0, \bar{A}]$ *and any* $w \geq 0$,

$$\psi(A, w) = \varphi(A, k_r^*(w)). \tag{3.7}$$

(iii) For any $A > 0$, *function* $\lambda(A, \cdot)$ *is the inverse of the function* $\ell \longmapsto A \cdot F_\ell(1, \ell)$ *and, moreover* $\lambda(0, w) = 0$ *if* $w > 0$.

(iv) For any $A > 0$ *and* $w \geq 0$,

$$\rho\left(w\right) = F_k\left(k_r^*(w), \bar{L}\right). \tag{3.8}$$

(v) For any $w \geq 0$, *the triple* $(c_1^*(w), k_s^*(w), k_r^*(w))$ *maximizes*

$$u(c_1) + \int v\left(r \cdot k_s + A \cdot \rho\left(w\right) \cdot k_r\right) dP(A) \tag{3.9}$$

under the constraint

$$c_1 + k_s + k_r = e + w\bar{L}. \tag{3.10}$$

Condition (3.2) implies that the wage process $\{\tilde{w}_t\}$ is a time-homogenous Markov process. Condition (3.3) and statement (v) imply that, for any t, the triple $(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t)$ that is chosen by generation t in period t depends only on the wage rate $\tilde{w}_t$ that this generation encounters in the market. Condition (3.4) and statement (iii) determine this generation's demand for labour $\tilde{\ell}_r^t$ in period $t+1$ as a function of the pair $(\tilde{A}_{t+1}, \tilde{w}_{t+1})$ and the capital investment $\tilde{k}_r^t$ in period t. Condition (3.5) determines this generation's consumption in period $t+1$ as a sum of the returns on safe and risky investments $\tilde{k}_s^t$ and $\tilde{k}_r^t$ in period t. As indicated in statement (iv), the equilibrium rate of return $\tilde{A}_{t+1} \cdot \rho\left(\tilde{w}_t\right)$ on the risky investment in period t depends on the productivity parameter $\tilde{A}_{t+1}$ and the investment $k_r^*(w)$ in period t.

To see the underlying logic, notice that, in any period t, the labour demand decision of the old generation is taken under full information about the realization of the productivity parameter $\tilde{A}_t$. In any best-response plan, therefore, $\tilde{\ell}_r^{t-1}$ must satisfy the first-order condition

$$\tilde{A}_t F_\ell(\tilde{k}_r^{t-1}, \tilde{\ell}_r^{t-1})) = \tilde{w}_t. \tag{3.11}$$

Together with the market-clearing condition (2.15), (3.11) implies that $\tilde{w}_t = \varphi(\tilde{A}_t, \tilde{k}_r^{t-1})$ for all t. For $t = 1$, this yields condition (3.1), for $t > 1$, it yields condition (3.2), which is derived by setting $\tilde{k}_r^{t-1} = k_r^*(\tilde{w}_{t-1})$.

Because F exhibits constant returns to scale, (3.11) can also be written in the form (3.4) where $\lambda(\cdot)$ has the form given in statement (ii). Thus people born in period t plan their labour demands in period $t+1$ to take the form (3.4). Their budget constraint for period $t+1$ can therefore be rewritten as

$$\tilde{c}_2^t = r\tilde{k}_s^t + \tilde{A}_{t+1} F\left(1, \lambda\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right)\right) \cdot \tilde{k}_r^t - \tilde{w}_{t+1} \cdot \lambda\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right) \cdot \tilde{k}_r^t. \quad (3.12)$$

By the definition of $\lambda(\cdot)$ and the constant-returns-to-scale assumption, this simplifies to

$$\tilde{c}_2^t = r\tilde{k}_s^t + \tilde{A}_{t+1} \cdot \hat{\rho}\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right) \cdot \tilde{k}_r^t, \quad (3.13)$$

where

$$\hat{\rho}\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right) = F_k(1, \lambda\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right)). \quad (3.14)$$

In equilibrium, $\lambda\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right)$ is the same for all participants, and period $t+1$ labour market clearing implies

$$F_k(1, \lambda\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right)) = F_k(\tilde{k}_r^t, \bar{L}) = F_k(k_r^*(\tilde{w}_t), \bar{L})$$

and therefore,

$$\hat{\rho}\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right) = F_k(k_r^*(\tilde{w}_t), \bar{L}) = \rho\left(\tilde{w}_t\right) \quad (3.15)$$

for $\rho(\cdot)$ given by statement (iv). Conditional on the event $\tilde{w}_t = w$, the maximization problem of a person born in period t is therefore equivalent to the problem of maximizing (3.9) subject to (3.10), which is just statement (v).

In this analysis, the variable $\tilde{k}_r^t$ plays a dual role. On the one hand, $\tilde{k}_r^t$ is an important element in the optimization of any one person that is born in period t. On the other hand, the aggregate $N \cdot \tilde{k}_r^t$ determines the aggregate demand for labour

$$N \cdot \tilde{\ell}_r^t = \lambda\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right) \cdot N \cdot \tilde{k}_r^t$$

in period $t+1$ and therefore the equilibrium wage rate $\tilde{w}_{t+1}$, as well as the state-contingent rate of return $\hat{\rho}\left(\tilde{A}_{t+1}, \tilde{w}_{t+1}\right)$ on risky investments in period t.

People are assumed to behave as price takers, so a person born in period t chooses the risky investment $\tilde{k}_r^t$ without taking account of the effect that this choice may have on the wage rate $\tilde{w}_{t+1}$. As in other models, the price taking assumption is justified if the effect of any person's individual choice is actually negligible, e.g., if we have a continuum of individuals (of total mass N), so that no one individual has a noticeable impact on aggregates.[9]

The hard part of the proof of Proposition 3.1 consists in showing that these two roles of the variable $\tilde{k}_r^t$ are consistent with each other so that period t expectations of state-contingent wage rates $\tilde{w}_{t+1}$ will induce investments $\tilde{k}_r^t$ in

[9] If the number of individuals is finite but large, arguments from the literature on Cournot convergence can be used to show that price taking is at least approximately optimal because the actual influence of an individual on prices (wage rates) is small, see, e.g., Mas-Colell et al. (1995), Ch. 12.F.

period t that will cause the actual wage rates in period $t+1$ to confirm those expectations. In this consistent feedback loop, the wage rate $\tilde{w}_{t+1}$ depends only on $\tilde{A}_{t+1}$ and $\tilde{w}_t$ and is otherwise independent of prior events.

Under the given assumptions, for any w, there is only one such consistent feedback loop that starts from the wage rate $\tilde{w}_t = w$. If there were several, selection of a consistent feedback loop might depend on previous history or on extraneous shocks such as sunspots. In this case, there would still be equilibria with the time-homogeneous Markov structure specified in Proposition 3.1, but there might also be other equilibria.

The result that, for any w, there is only one such consistent feedback loop that starts from the wage rate $\tilde{w}_t = w$ is based on the assumed monotonicity of the functions $c \longmapsto -\frac{v''(c)}{v'(c)}$ and $k_r \mapsto k_r F_k(k_r, \ell_r)$. Without these monotonicity properties, I cannot rule out the possibility that, for any w, there might be several consistent feedback loops that start from the wage rate $\tilde{w}_t = w$. With several consistent feedback loops, the feedback loop that pertains to the equilibrium might be conditioned on past history, or even on irrelevant variables such as the activity of sunspots, so that the equilibrium would not reflect the recursive structure of the model.

3.2 Qualitative Properties

Proposition 3.1 indicates that the maximization problem in statement (v) provides the key to understanding the economics of the model. The following proposition lists some of the properties of the solutions to this problem.

Proposition 3.2 *The functions $c_1^*(\cdot), k_s^*(\cdot), k_r^*(\cdot)$ in Proposition 3.1 are continuous and also have the following properties.*

(a) If $e+\underline{A}>0$, there exists $\underline{c}>0$ such that for all $w \geq 0$, $c_1^*(w) \geq \underline{c}$.

(b) If $P(\{0\}) > 0$, $k_s^*(w) > 0$ for all $w \geq 0$. If $P([0, \frac{r}{\rho(w)})) = 0$, $k_s^*(w) = 0$ for all $w \geq 0$.

(c) If $e+\underline{A}>0$, there exist $\underline{K}>0$ and $\bar{K}>\underline{K}$ such that $\underline{K} \leq k_r^*(w) \leq \bar{K}$ for all $w \geq 0$. Moreover, the function $w \longmapsto k_r^*(w)$ is nondecreasing.

(d) For all $w \geq 0$, the equilibrium expected rate of return on risky investments, $A^* \cdot \rho(w)$, is strictly greater than r, the rate of return on safe investments.

Statement (a) holds because $u'(0) = \infty$ and $e + \underline{A} > 0$. The positivity of $e+\underline{A}$ ensures that, for any t and any value of $\tilde{A}_t$, a young person in period t has a positive budget. Because $u'(0) = \infty$, some of that budget is used for current consumption.

The first part of statement (b) is based on a similar logic: Because $v'(0) = \infty$, fear of the possibility that $\tilde{A}_{t+1}$ might be zero induces positive investment in the safe asset, even if r is very small. The second part of statement (b) asserts that no investment in the safe asset is made if the safe asset is dominated by the risky asset.

To understand statements (c) and (d), notice that, by standard considerations of portfolio choice, investment in the risky asset is strictly positive if and only if the expected rate of return $\rho(w) \cdot \int AdP(A)$ on the risky asset is greater than r. If $k_r^*(w)$ were very large, then, by (3.8) and (2.1), $\rho(w)$ would be less than r, implying that $k_r^*(w) = 0$, a contradiction. If $k_r^*(w)$ were close to zero, then, by (3.8) and (2.1), $\rho(w)$ would be very large; in this case, the maximization of (3.9) subject to (3.10) would require that $k_r^*(w) \cdot \rho(w)$ be very large, which is incompatible with the feasibility condition that $k_r^*(w) \cdot \rho(w) \leq F(k_r^*(w), \bar{L})$ (for $k_r^*(w)$ close to zero).

Finally the monotonicity of the function of $w \longmapsto k_r^*(w)$ follows from the assumption of nonincreasing absolute risk aversion.

3.3 Stochastic Stability

The behaviour of the economy over time is driven by the wage process $\{\tilde{w}_t\}_{t=1}^{\infty}$. By (3.2), (3.6), and (3.7), this process can be written in the form

$$\tilde{w}_{t+1} = \tilde{A}_{t+1} \cdot F_\ell(k_r^*(\tilde{w}_t), \bar{L}). \tag{3.16}$$

This equation specifies a time-homogeneous Markov process. A natural question to ask is whether the probability distributions G_t of the state-dependent wage rates $\tilde{w}_t$ converge to some invariant probability distribution G^* as t goes out of bounds.

I use the approach of Hopenhayn and Prescott (1992) to study this question. They give three sufficent conditions for the convergence of the probability distributions G_t to an invariant distribution G^* : Stochastic monotonicity, uniform boundedness and what they call a *monotone mixing condition.* To see that the process given by (3.16) satisfies stochastic monotonicity and uniform boundedness (as well as continuity), it suffices to observe that the map $w \longmapsto A \cdot F_\ell(k_r^*(w), \bar{L})$ is nondecreasing, continuous, and bounded. The reason is that, by Proposition 3.2, the map $w \longmapsto k_r^*(w)$ is nondecreasing, continuous, and bounded, as well as bounded away from zero, and, by the constant-returns-to-scale property and the continuous differentiability and quasi-concavity of F, the function $k \longmapsto F_\ell(k, \bar{L})$ is also nondecreasing and continuous.

The *monotone mixing condition* requires that, if $\bar{w}$ and $\underline{w}$ are the uniform upper and lower bounds on the random variables $\tilde{w}_t$, $t = 1, 2, ...$, there exist $w^{**} \in (\underline{w}, \bar{w})$ and T, such that

$$\Pr(\{\tilde{w}_T \leq w^{**}\} | \tilde{w}_1 = \bar{w}) > 0$$

and

$$\Pr(\{\tilde{w}_T \geq w^{**}\} | \tilde{w}_1 = \underline{w}) > 0.$$

One easily verifies that this condition is satisfied if there exist $\varepsilon > 0$ and T, such that the solution to the difference equation

$$\hat{w}_{t+1} = \underline{A} \cdot F_\ell(k_r^*(\hat{w}_t), \bar{L})$$

with the initial condition $\hat{w}_0 = \bar{w}$ satisfies $\hat{w}_T \leq \underline{w} + \varepsilon$. This latter condition is satisfied with $T = 1$ if

$$\frac{\underline{A}}{\bar{A}} < \frac{F_\ell(\underline{K}, \bar{L})}{F_\ell(\bar{K}, \bar{L})}. \tag{3.17}$$

and, in particular, if $\underline{A} = 0$.

Proposition 3.3 *For any equilibrium, there exists an invariant distribution G^* for the wage process $\{\tilde{w}_t\}_{t=0}^{\infty}$. If, in addition to stochastic monotonicity and boundedness, the wage process $\{\tilde{w}_t\}_{t=0}^{\infty}$ satisfies the monotone mixing condition, the invariant distribution G^* is unique, and, regardless of initial conditions, the probability distribution G_t of the wage rate $\tilde{w}_t$ converges to G^* as t goes out of bounds.*

Proposition 3.3 provides a basis for reducing the dependence of welfare assessments on the information that people have when they take their decisions. At any time t, generation t knows the wage rate $\tilde{w}_t$ and the mean rate of return $E\tilde{A}_{t+1} \cdot \rho(\tilde{w}_t)$ on risky assets. This information can restrict the scope for *interim* Pareto improvements, i.e. Pareto improvements relative to the positions indicated by this information. From an *ex ante* perspective, however, they may not matter.

From an *ex ante* perspective, welfare assessments of changes in allocations might still be affected by initial conditions, in particular, the amount k_r^0 of risky capital that generation brings to period 1. However, if the probability distributions G_t converge to G^* as t goes out of bounds, the impact of initial conditions disappears over time and, for very large t, the *ex ante* expected utility of a person born in period t is approximately the same as the expected value

$$\int [u(c_1^*(w)) + v(rk_s^*(w) + A\rho(w)k_r^*(w))]dG^*(w) \tag{3.18}$$

of a person's utility from the triple $(c_1^*(w), k_s^*(w), k_r^*(w))$ with respect to the distribution G^*.

The convergence in Proposition 3.3 is not just a matter of probability distributions. By the ergodic theorem for stationary Markov processes,[10] the convergence also applies to the distributions that are obtained from time averages along sample paths.

4 Welfare Assessments

Before proceeding with welfare assessments of equilibrium allocations, I note that in the present context, the standard definition of Pareto efficiency is unsatisfactory. According to this definition, a feasible allocation is Pareto-efficient

[10] See, e.g., Theorem 6.1, p. 219, in Doob (1953).

if and only if there is no other feasible allocation under which no participant is worse off and some participants are strictly better off than under the original allocation. This definition is imprecise because it does not make clear whether people assess the change in question from an *ex ante* perspective or from an *interim* perspective, conditioning on the information they have received.

I will actually consider both, beginning with an *interim* perspective where each generation t assesses a change of allocation on the basis of the information available to them, in particular the information about the productivity parameters $\tilde{A}_1, ..., \tilde{A}_t$ and the associated wage rates $\tilde{w}_1, ..., \tilde{w}_t$. Thus, one allocation is *interim* Pareto-preferred to another if, conditioning on the information that is available to agents when they take their decisions and regardless of the value that information may take, no participant is worse off and some participants are strictly better off under the first allocation than under the second allocation.

In contrast, one allocation is *ex ante* Pareto-preferred to another if, without any conditioning, the *ex ante* expected utility of every participant is at least as high, and of some participants strictly higher, under the first than under the second allocation. The *ex ante* Pareto criterion is less restrictive than the *interim* Pareto criterion: If one allocation is *interim* Pareto-preferred to another, then it is also *ex ante* Pareto-preferred, but the converse is not necessarily true.

From an *ex ante* perspective, there is no reason to expect equilibrium allocations to be Pareto efficient unless some constraints are imposed on admissible reallocations. The reason is that equilibrium allocations will usually involve inefficient risk sharing. Risks that are due to the dependence of output on the productivity parameter $\tilde{A}_t$ are shared between generations $t-1$ and t in proportion to the capital shares and labour shares in the output. Efficient risk sharing however would require that these risks be shared in proportion to the respective degrees of risk tolerance that are inherent in the utility functions $u(\cdot)$ and $v(\cdot)$.[11] There is no reason why risk sharing according to factor shares should accord with the latter criterion.

To avoid the issue of efficient sharing of productivity risks, at least for now, I limit the analysis to feasible reallocations that involve only non-contingent changes in consumption levels. I do not allow for contingent changes in consumption levels or for changes in investment levels. Thus, a feasible allocation $(\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ is *constrained interim* (*constrained ex ante) efficient* if and only if there exists no sequence $\{\Delta_t\}_{t=1}^{\infty}$ such that the allocation $(\tilde{\ell}_r^0, \tilde{c}_2^0 + \Delta_1), \{(\tilde{c}_1^t - \Delta_t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t + \Delta_{t+1})\}_{t=1}^{\infty}$ is *interim* (*ex ante*) Pareto-preferred to the original allocation.[12]

Proposition 4.1 *Let* $(\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ *be an equilibrium allocation and assume that* $\tilde{k}_s^t > 0$ *for all* t*. If* $r > 1$*, the allocation is constrained interim Pareto-efficient. If* $r < 1$*, the allocation is not constrained interim efficient.*[13]

[11] The degree of risk tolerance is equal to the inverse of the degree of absolute risk aversion.

[12] The concept of constrained efficiency was introduced by Diamond (1967) and Hart (1975).

[13] Given the analysis of Okuno and Zilcha (1980), I conjecture that, if $r = 1$, the allocation is constrained interim Pareto efficient, but the argument seems too involved to be worth pursuing

Proposition 4.1 provides the simplest version of the claim that, in the presence of both safe and risky assets, the efficiency of *laissez-faire* allocations depends on whether or not the safe rate of return is greater or smaller than the growth rate. With a stationary population and with $\tilde{k}_s^t > 0$ for all t, this comparison hinges on whether $r > 1$ or $r < 1$. The (conditionally) expected rate of return, $E\tilde{A}_{t+1} \cdot \rho(\tilde{w}_t)$, plays no role. To be sure, if $E\tilde{A}_{t+1} \cdot \rho(\tilde{w}_t) < 1$, then, by part (d) of Proposition 3.2, we also have $r < 1$ so the equilibrium allocation is not constrained interim efficient. However, the very inequality $E\tilde{A}_{t+1} \cdot \rho(\tilde{w}_t) > r$ that is obtained from Proposition 3.2, implies that we can have $E\tilde{A}_{t+1} \cdot \rho(\tilde{w}_t) > 1$ and $r < 1$, in which case the equilibrium allocation fails to be constrained *interim* efficient.

The argument for this result is straightforward: The assessment of constrained interim efficiency depends on how the envisioned change affects the expected value of the utility $u(\tilde{c}_1^t) + v(\tilde{c}_2^t)$ of a person born in period t, conditional on the information available in period t, i.e., conditional on the wage rate $\tilde{w}_t$. For $\tilde{w}_t = w$, the value of this conditional expectation is equal to

$$u(c_1^*(w)) + \int v(rk_s^*(w) + A\rho(w)k_r^*(w))dP(A). \tag{4.1}$$

For small $\Delta > 0$ and $\Delta_t = \Delta_{t+1} = \Delta$, the induced change in the conditionally expected utility is approximately equal to

$$-\Delta u'(c_1^*(w)) + \Delta \int v'(rk_s^*(w) + A\rho(w)k_r^*(w))dP(A). \tag{4.2}$$

By the first-order conditions for the choice of $c_1^*(w)$ and $k_s^*(w) > 0$,

$$u'(c_1^*(w)) = r \int v'(rk_s^*(w) + A\rho(w)k_r^*(w))dP(A), \tag{4.3}$$

so (4.2) can be rewritten in the form

$$-\Delta u'(c_1^*(w)) + \Delta \frac{1}{r} u'(c_1^*(w)), \tag{4.4}$$

which is positive, regardless of w, if $r < 1$. The second statement in the proposition follows immediately. The proof of the first statement in the proposition is more involved; in this case, (4.3) implies that (4.2) is negative, but that leaves open the question whether Pareto improvements might be obtained in some other way. The full proof in the appendix shows that that is not the case.

In Proposition 4.1, the assumption that $\tilde{k}_s^t > 0$ for all t is problematic because it involves endogenous variables. By part (b) of Proposition 3.2, however, this assumption is always satisfied if there is a positive probability that the productivity shock might take the value zero. This observation yields the following corollary to Proposition 4.1.

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here.

Corollary 4.2 *Assume that $P(\{0\}) > 0$ and let $(\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ be an equilibrium allocation. If $r > 1$, the allocation is constrained interim Pareto-efficient. If $r < 1$, the allocation is not constrained interim efficient.*

If $P(\{0\}) = 0$, there may be periods and contingencies in which no safe investments are made. In such instances, the rate of return r on the safe asset cannot be interpreted as an intertemporal relative price. An implicit intertemporal relative price is given by the variable

$$R(\tilde{w}_t) := \frac{u'(c_1^*(\tilde{w}_t))}{Ev'(rk_s^*(\tilde{w}_t) + \tilde{A}_{t+1}\rho(\tilde{w}_t)k_r^*(\tilde{w}_t))}, \tag{4.5}$$

which corresponds to the marginal rate of substitution of a person born in period t between a change in first-period consumption and an equal change in second-period consumption when both changes can only depend on information available at t.[14] In the equilibria of Proposition 4.1 and Corollary 4.2, $R(\tilde{w}_t)$ is always equal to r, but if there are periods and contingencies where $k_s^*(\tilde{w}_t) = 0$, $R(\tilde{w}_t)$ can exceed r. The following result generalizes Proposition 4.1 to allow for this possibility.

Proposition 4.3 *Let $\{\tilde{w}_t\}_{t=1}^{\infty}, (\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ be an equilibrium, assume that the wage process $\{\tilde{w}_t\}_{t=0}^{\infty}$ satisfies the monotone mixing condition and let G^* be the invariant probability distribution for this process. If $R(w) > 1$ for some w in the support of G^*, the equilibrium allocation is constrained interim Pareto-efficient. If $R(w) < 1$ for all w in the support of G^*, the allocation is not constrained interim efficient.*

The variability of the marginal rates of substitution $R(\tilde{w}_t)$ over periods and contingencies introduces an asymmetry into the assessment of dynamic inefficiency. For an assessment of efficiency, it is enough that $R(w) > 1$ on *some* nonnegligible set of wage rates. Conversely, an assessment of dynamic inefficiency requires that $R(w) \leq 1$ for *all* relevant wage rates.

The asymmetry is due to the fact that, under a criterion of *interim* efficiency, participants have a lot of veto powers. An *interim* Pareto improvement must make the participants better off, regardless of what the value of the current state variable $\tilde{w}_t$ may be. This veto power explains the asymmetry of the quantifiers "some" and "all" in the preceding paragraph and in the two parts of Proposition 4.3.

Abel et al. (1989) give a different sufficient condition for interim efficiency of *laissez-faire* allocations in an overlapping-generations model. Their condition is ostensibly not about rates of return but about cash flows between the consumption side and the production side of the economy. An equilibrium

[14] Blanchard (2019) introduced this approach to specifying the riskless rate in the absence of a riskless asset.

$\{\tilde{w}_t\}_{t=1}^{\infty}, (\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ satisfies their *net dividend criterion for efficiency* if there exists $\varepsilon > 0$ such that, for all t,

$$\tilde{c}_2^{t-1} \geq (1+\varepsilon)(\tilde{k}_r^t + \tilde{k}_s^t) \tag{4.6}$$

almost surely.[15] Thus the consumption $\tilde{c}_2^{t-1}$ of the old in period t must exceed the new investment $\tilde{k}_r^t + \tilde{k}_s^t$ of the young in period t by a fraction ε. The difference $\tilde{c}_2^{t-1} - (\tilde{k}_r^t + \tilde{k}_s^t)$ is the *net* payment flow from the production side to the consumption side of the economy; hence the term "net dividend".[16]

Condition (4.6) should not be misunderstood as a rate-of-return condition. A rate-of return condition would compare the payout $\tilde{c}_t$ to the *old* investment $\tilde{k}_r^{t-1} + \tilde{k}_s^{t-1}$. However, the followng result shows that, for some values of the state variable $\tilde{w}_{t-1}$ in period $t-1$ that belong to the support of the invariant distribution G^*, condition (4.6) actually implies the inequality $R(\tilde{w}_t) > 1$, so the result of Abel et al. (1989) is actually a special case of Proposition 4.3. The proof makes essential use of the stationarity properties of the equilibrium wage process $\{\tilde{w}_t\}_{t=0}^{\infty}$ and the equilibrium capital process $\{\tilde{k}_r^t\}_{t=0}^{\infty}$.[17]

Proposition 4.4 *Let $\{\tilde{w}_t\}_{t=1}^{\infty}, (\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ be an equilibrium, assume that the wage process $\{\tilde{w}_t\}_{t=0}^{\infty}$ satisfies the monotone mixing condition and let G^* be the invariant probability distribution for this process. If the equilibrium satisfies the net dividend criterion for efficiency, then $R(w) > 1$ for w close to the minimum of the support of G^*.*

Corollary 4.5 *Under the assumptions of Proposition 4.4, the equilibrium allocation is constrained interim efficient.*

I next turn to *ex ante* efficiency.

[15] Abel et al. (1989) also have a *net dividend criterion for dynamic inefficiency*, essentially (4.6) with $\leq$ instead of $\geq$ and $(1-\varepsilon)$ instead of $(1+\varepsilon)$. This criterion yields a failure of *unconstrained* interim efficiency, with Pareto improvements from reductions of all capital investments by a constant multiple δ.

[16] If the production function has the Cobb-Douglas form $F(k,\ell) = k^{\alpha}\ell^{1-\alpha}$, the net dividend criterion is necessarily satisfied if $\alpha > \frac{1}{2}$. In this case, the budget constraint for generation t implies $\tilde{k}_r^t + \tilde{k}_s^t \leq \tilde{w}_t$. Moreover, in the Cobb-Douglas case, the wage equation (3.2) takes the form $\tilde{w}_t = (1+\varepsilon)\alpha\tilde{A}_t(\tilde{k}_r^{t-1})^{\alpha}\bar{L}^{1-\alpha}$, where $\varepsilon = \frac{1-2\alpha}{\alpha}$. Hence $\tilde{k}_r^t + \tilde{k}_s^t \leq (1+\varepsilon)[\alpha\tilde{A}_t\rho(\tilde{w}_{t-1})\tilde{k}_r^{t-1} + r\tilde{k}_s^{t-1}] = \tilde{c}_2^t$.

[17] The claim of Abel et al. (1989) and the argument they make for this claim do not invoke stationarity. However, the argument is flawed, and I do not see a way to repair it. Their proof strategy is to show that equilibrium allocations satisfy the first-order conditions for the solutions to a certain welfare maximization problem. However, as was pointed out by Chattopadhyay (2008), the argument of Abel et al. (1989) is invalid if the welfare function is unbounded. Chattopadhyay gives examples where the claim of Abel et al. (1989) is false. These examples involve nonstationarity of equilibrium capital processes.

Proposition 4.6 *Let* $\{\tilde{w}_t\}_{t=1}^{\infty}, (\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ *be an equilibrium, assume that the wage process* $\{\tilde{w}_t\}_{t=0}^{\infty}$ *satisfies the monotone mixing condition and let* G^* *be the invariant probability distribution for this process. If* $\int u'(c_1^*(w)) \cdot \frac{R(w)-1}{R(w)}\ dG^*(w) > 0$*, the equilibrium allocation is constrained ex ante Pareto-efficient. If* $\int u'(c_1^*(w)) \cdot \frac{R(w)-1}{R(w)}\ dG^*(w) < 0$*, the equilibrium allocation is not constrained ex ante efficient.*

In this proposition, the assessment of dynamic inefficiency hinges on whether a marginal-utility-weighted average of the inverse of marginal rate of substitution $R(w)$ is greater or less than one. The average is taken with respect to the invariant distribution G^*. If the inequality in the first statement or the inequality of the second statement of the proposition holds, then, by Proposition 3.3, the corresponding inequality also holds if the average is taken with respect to the distribution G_t of the random variable $\tilde{w}_t$ for t sufficiently large.

In an *ex ante* assessment of Pareto efficiency, expression (4.1) for the expected utility of a person born in period t under the initial allocation conditionally on the event $\tilde{w}_t = w$ is replaced by the *ex ante* expectation

$$\int \left[u(c_1^*(w)) + \int v(rk_s^*(w) + A\rho(w)k_r^*(w))dP(A) \right] dG_t(w).$$

For small $\Delta > 0$ and $\Delta_t = \Delta_{t+1} = \Delta$, therefore expressions (4.2) and (4.4) now take the form

$$-\Delta \int u'(c_1^*(w))dG_t(w) + \Delta \int \int v'(rk_s^*(w) + A\rho(w)k_r^*(w))dP(A)dG_t(w)$$

and

$$-\Delta \int u'(c_1^*(w))dG_t(w) + \Delta \int u'(c_1^*(w)) \frac{1}{R(w)} dG_t(w), \tag{4.7}$$

which is positive or negative depending on whether the marginal-utility weighted average of $\frac{1}{R(w)}$ under the distribution G_t is greater than one or less than one. If this average is less than zero under the invariant distribution G^*, it is also negative under the distribution G_t for t exceeding some bound t^*, and the new allocation that is generated by a sequence $\{\Delta_t\}_{t=1}^{\infty}$ with $\Delta_t = 0$ for $t = 1, ...t^* - 1$ and $\Delta_t = \Delta > 0$ for $t \geq t^*$, for sufficiently small Δ and suitably chosen t^* provides for an *ex ante* Pareto improvement over the original allocation.[18]

As in the case of Proposition 4.1, in Propositions 4.3 and 4.6, the rates of return on risky investments play no role. Only the "riskless" intertemporal marginal rates of substitution $R(w)$ matter.

In making the contrary case, Abel et al. (1989, p. 14) argue that arguments used to demonstrate dynamic inefficiency may not be available because a

[18] The date t^* at which the intervention begins can be set equal to 1 if the levels k_s^0, k_r^0 of safe and risky capital held by generation 0 at $t = 1$ are seen as the realizations of random variables $\tilde{k}_s^0 = k_s^*(\tilde{w}_0), \tilde{k}_r^0 = k_r^*(\tilde{w}_0)$, with $\tilde{w}_0$ distributed as G^*.

person born in period t may not have the resources required for the stipulated change $-\Delta_t$ in the person's consumption level. In the preceding analysis, I have eliminated this possibility through the assumption that each person receives an initial endowment $e > 0$ in addition to whatever the wage income $\tilde{w}_t \bar{L}$ may be. Alternatively, I might have assumed that $\underline{A}$, the minimum of the support of the distribution P, is strictly positive. In this case, statement (a) of Proposition 3.2 would still be valid, i.e., first-period consumption $c_1^*(w)$ would still be bounded away from zero.

The objection of Abel et al. (1989) is relevant if $e = \underline{A} = 0$ so that $\tilde{w}_t$ and $c_1^*(\tilde{w}_t)$ take values arbitrarily close to zero. In this case, the condition that reallocations involve only non-contingent changes in consumption levels rules out any positive Δ.

The objection might be addressed by allowing changes in consumption levels to be made contingent on incomes. For example, one might replace the non-contingent Δ by some contingent

$$\Delta(w) = \Delta_0 \cdot \frac{1}{u'(c_1^*(w))} \tag{4.8}$$

for some $\Delta_0 > 0$. This specification would eliminate the mechanical problem posed by Abel et al. (1989).[19] It would also imply replacing the marginal-utility-weighted means $\int u'(c_1^*(w)) \cdot \frac{R(w)-1}{R(w)}\, dG^*(w)$ in Proposition 4.6 by the unweighted means $\int \frac{R(w)-1}{R(w)}\, dG^*(w)$ as (4.7) would take the form

$$-\Delta_0 \int dG_t(w) + \Delta_0 \int \frac{1}{R(w)} dG_t(w). \tag{4.9}$$

However, the specification (4.8) would not generally provide for *interim* Pareto improvements, as people who benefit from high wage rates are asked to make relatively larger sacrifices when they are young, for which they may not receive sufficient compensation when they are old. If $\int \frac{R(w)-1}{R(w)}\, dG^*(w) < 1$, the gains to people who face low wage rates more than make up for those losses to people who face high wage rates, so, from an *ex ante* perspective, the intervention makes for a Pareto improvement. But the *ex ante* approach to welfare assessments raises many more questions about the scope for efficient sharing of the risks arising from the productivity parameters $\tilde{A}_t$.[20]

5 Pareto Improvements by Fiscal Interventions

So far I have considered the scope for Pareto improvements by means of direct interventions in consumption allocations, without allowing for any behavioural

[19] Abel et al. (1989) themselves use contingent interventions for their result on dynamic inefficiency. As mentioned in fn. 15, they rely on *equal multiplicative reductions* of capital investments of young people in all periods to raise old people's consumption in all periods. These reductions depend on *laissez-faire* investment levels and therefore on the contingencies that determine these investment levels.

[20] Some of these issues are discussed in Ball and Mankiw (2007).

effects that the interventions might have. I now turn to the question whether *laissez-faire* allocations can be improved upon through fiscal policy. Fiscal policy does not by itself determine the overall allocation but leaves room for behavioural reactions of the participants. These behavioural reactions might have adverse effects that eliminate the scope for Pareto improvements that would otherwise be there.

Blanchard (2019) considers the scope for Pareto improvements through lump sum taxes and transfers. These taxes and transfers correspond to the Δ's in the preceding analysis, meaning that a young person in some period t makes a lump sum payment Δ_t in that period and receives a lump sum transfer Δ_{t+1} in the next period. However, whereas the preceding analysis assumed that the changes in consumption levels due to these taxes and transfers represent the *only* departures from the *laissez-faire* equilibrium allocation, Blanchard (2019) also allows for adjustments of behaviours in response to the policy intervention. Some behaviour adjustments do not matter for welfare assessments because, at the *laissez-faire* equilibrium allocation, at the margin, the people concerned are indifferent. However, some behaviour adjustments can have Pareto-relevant external effects.

For suppose that people change their investments in reaction to the tax imposed in the first period of their lives and in anticipation of receiving a transfer in the second period of their lives. In the model of this paper, a change in the level of safe investments has no further repercussions, but a change in the level of risky investments affects labour market outcomes in the next period. If the members of generation t reduce their investments in risky capital, the members of generation $t + 1$ will face lower wage rates and earn less.

Blanchard considers the possibility that such harm to the members of generation $t+1$ might be outweighed by the benefits that these people in turn obtain from reducing their own capital investments, causing market-clearing wage rates in period $t+2$ to be lower and the rates of return on their own risky investments to be higher than without the policy intervention. In Section 6 below and in Appendix B, however, I will argue that there is no hope for an unambiguous assessment of this tradeoff. The reason is that the change in the wage rate in period $t + 1$ depends on the value of the productivity parameter $\tilde{A}_{t+1}$. If the value of this productivity parameter is large, the welfare effect of the change in the wage rate $\tilde{w}_{t+1}$ exceeds the (conditionally expected) welfare effect of the change in the rate of return $\tilde{A}_{t+2} \cdot \rho(\tilde{w}_{t+1})$; if the value of $\tilde{A}_{t+1}$ is small, the tradeoff goes the other way.

Even from an *ex ante* perspective, taking expectations with respect to the value of $\tilde{A}_{t+1}$, the welfare assessment of these price effects is unclear. Given the incompleteness of the market system, there is no presumption that the *laissez-faire* allocation of these risks is efficient, nor is there any presumption as to the kind of correction that the inefficiency of the risk allocation might call for.

Fiscal interventions can however be designed in such a way that price effects play no role. For this purpose, I consider a combination of a tax-and-transfer scheme with a specific subsidy to risky capital. Thus, for each t, a young person born in period t pays a lump sum tax Δ^t in the first period of his or her life and receives a lump sum subsidy $\hat{\Delta}^t$ in the second period, together with a specific subsidy $\sigma^t \cdot k_r^t$ according to the person's investment in risky capital in period t. (The subsidy can be negative, i.e., a tax.) I allow the subsidy $\hat{\Delta}^t$ and the subsidy rate σ^t to depend on the wage rate w_t that prevails in period t.[21]

I do *not* allow $\hat{\Delta}^t$ and σ^t to depend on the wage rate w_{t+1} that prevails in period $t+1$. The reason is that I again want to abstract from the issue of how generations t and $t+1$ share the risks attached to the productivity parameter $\tilde{A}_{t+1}$ and thereby to the wage rate $\tilde{w}_{t+1}$. The question how this risk should be shared is interesting and important, but seems unrelated to dynamic inefficiency.

Given a *fiscal policy* $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ and given a wage process $\{\tilde{w}_t\}_{t=1}^{\infty}$, a person born in period $t \geq 1$ now chooses a plan $(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)$ to maximize the expected utility

$$E[u(\tilde{c}_1^t) + v(\tilde{c}_2^t)] \tag{5.10}$$

subject to the constraints

$$\tilde{c}_1^t + \tilde{k}_s^t + \tilde{k}_r^t = e - \Delta^t + \tilde{w}_t \bar{L}, \tag{5.11}$$

and

$$\tilde{c}_2^t = r\tilde{k}_s^t + \tilde{A}_{t+1} F(\tilde{k}_r^t, \tilde{\ell}_r^t) - \tilde{w}_{t+1}\tilde{\ell}_r^t + \hat{\Delta}^t(\tilde{w}_t) + \sigma^t(\tilde{w}_t) \cdot \tilde{k}_r^t. \tag{5.12}$$

As before, the plan $(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)$ must be adapted to the information available to the individual, i.e., thus $\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t$ can only depend on information available at t, and $\tilde{\ell}_r^t, \tilde{c}_2^t$ can only depend on information available at $t+1$.

Similarly, an old individual in period 1 chooses $(\tilde{\ell}_r^0, \tilde{c}_2^0)$ to maximize the utility $v(\tilde{c}_2^0)$ under the constraint

$$\tilde{c}_2^0 = rk_s^0 + \tilde{A}_{t+1} F(k_r^0, \bar{L}) - \tilde{w}_1 \tilde{\ell}_r^0 + \hat{\Delta}^0 + \sigma^0 \cdot \tilde{k}_r^t, \tag{5.13}$$

where $\hat{\Delta}^0$ and σ^0 are treated as constants and any dependence on a previously prevailing wage rate is ignored.

As before, optimal plans are called *best responses*, now to the fiscal policy $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ as well as the process $\{\tilde{A}_t, \tilde{w}_t\}_{t=1}^{\infty}$ of random productivity parameters and wage rates. A s*ymmetric equilibrium for the fiscal policy* $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ and the productivity parameter process $\{\tilde{A}_t\}$ is given by a wage process $\{\tilde{w}_t\}_{t=1}^{\infty}$ and a feasible allocation $(\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ such that, for any $t \geq 1$, the plan $(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)$ for a person born in period t is a best response of that person to the fiscal policy $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ and the process $\{\tilde{A}_t, \tilde{w}_t\}_{t=1}^{\infty}$ of productivity parameters and wage rates and so is the plan $(\tilde{\ell}_r^0, \tilde{c}_2^0)$ for an old person in period 1. The fiscal policy $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ is said to be *feasible* if there exists an equilibrium for this fiscal policy and the process $\{\tilde{A}_t\}$ such that, in this equilibrium, the government budget constraints

$$\hat{\Delta}^0 + \sigma^0 \cdot k_r^0 \leq \Delta^1 \tag{5.14}$$

[21] I use superscripts to indicate that these parts of the fiscal policy concern generation t. The second-period payments to people of this generation are of course made in period $t+1$.

and, for $t \geq 1$,

$$\hat{\Delta}^t(\tilde{w}_t) + \sigma^t(\tilde{w}_t) \cdot \tilde{k}_r^t \leq \Delta^{t+1} \tag{5.15}$$

are all satisfied. The specification (5.14), (5.15) of the government budget constraints presumes that the fiscal policy does not involve any administrative costs. This presumption is unrealistic, but helps focussing the analysis on essentials. The route to a generalization is straightforward.[22]

The following result shows that, if the condition for the failure of constrained interim efficiency of an equilibrium allocation under *laissez faire* that is given in Proposition 4.3 is satisfied, then there also exists a feasible fiscal policy such that, from an *interim* perspective, the associated change in the equilibrium allocation is a Pareto improvement. Given that Proposition 4.3 generalizes Proposition 4.1, this result also implies that, if under *laissez faire* $\tilde{k}_r^t > 0$ and, moreover $r < 1$, there is feasible fiscal policy that provides for a Pareto improvement from an *interim* perspective.

Proposition 5.7 *Let* $\{\tilde{w}_t\}_{t=1}^{\infty}, (\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ *be an equilibrium when there is no fiscal policy. For any* t*, let* $R(\tilde{w}_t)$ *be the equilibrium value of the marginal rate of substitution that is given by (4.5). If* $R(w) < 1$ *for all* $w \geq 0$*, there exists a feasible fiscal policy* $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ *such that the consumption allocation associated with this fiscal policy provides for an interim Pareto improvement over the consumption allocation in the absence of fiscal policy.*

Remark 5.8 *The Pareto-improving fiscal policy in Proposition 5.7 is time-independent, i.e., the triples* $(\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot))$ *are the same for all* t*. Moreover, the equilibrium associated with this fiscal policy involves the same processes* $\{\tilde{w}_t\}_{t=1}^{\infty}$ *and* $\{\tilde{k}_r^t\}_{t=1}^{\infty}$ *for wage rates and for risky investments as the original equilibrium in the absence of fiscal policy.*

The proposition combines two observations: First, the specific subsidy to risky capital can be used to eliminate any incentive effects that the fiscal intervention might have on investments in risky capital and thereby on subsequent wage rates. Second, given that the specific subsidy to risky capital eliminates the price effects that complicate Blanchard's analysis for risky assets, the arguments in the proofs of Propositions 4.1 and 4.3 and in Blanchard's (2019) analysis of direct effects of transfers, dominate the overall assessment of the policy intervention.

The Pareto-improving fiscal policy in Proposition 5.7 satisfies all government budget constraints as equations. This observation might give rise to the question whether participants would not appreciate the impact that their investments in risky capital have on the government budget and therefore, indirectly, on the government's ability to pay them a lump sum subsidy $\hat{\Delta}^t(\tilde{w}_t)$ in addition to the specific subsidy $\sigma^t(\tilde{w}_t) \cdot \tilde{k}_r^t$. The answer to this question is the same as

[22] In a different context, in Hellwig (2020/2021), I show how administrative costs of the government can be integrated into the analysis of dynamic inefficiency.

the answer to the question posed above about people's taking next period's wage rates as given when the wage rates depend on their own investments $\tilde{k}_r^t$ as given: Like the condition for period $t+1$ labour market clearing, the period $t+1$ government budget constraint depends on the aggregate (average) of generation t's investments in risky capital, and any one individual is too insignificant to affect this aggregate. The person who leaves $\tilde{k}_r^t$ at a relatively high level because the subsidy looks attractive discount the effect of this choice on the government's ability to pay the lump sum subsidy $\hat{\Delta}^t$ because this effect is in fact negligible when this is just one person in a large population.

For completeness, I note that there is also an analogue of Proposition 5.7 for *ex ante* improvements.

Proposition 5.9 *Let* $\{\tilde{w}_t\}_{t=1}^{\infty}, (\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ *be an equilibrium when there is no fiscal policy, assume that the wage process* $\{\tilde{w}_t\}_{t=0}^{\infty}$ *satisfies the monotone mixing condition and let* G^* *be the invariant probability distribution for this process. If* $\int u'(c_1^*(w)(\frac{R(w)-1}{R(w)})dG^*(w) < 0$, *there exists a feasible fiscal policy* $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ *such that the consumption allocation associated with this fiscal policy provides for an ex ante Pareto improvement over the consumption allocation in the absence of fiscal policy.*

6 Discussion

Why the Safe Rate? A key to understanding the results of this paper lies in the observation that, in Propositions 4.3 - 5.9, the relation of the "safe rate of interest" to the population growth rate is critical even when no safe assets are actually held. These propositions do not refer to the rate of return on safe assets, but to the implicit intertemporal relative price that is given by $R(\tilde{w}_t)$, the marginal rate of substitution between a change in first-period consumption and an equal change in second-period consumption when both changes can only depend on information available at t. This implicit intertemporal relative price matters because a direct intervention of the sort considered in Propositions 4.3 and 4.6, or a tax-and-transfer scheme of the sort considered in Propositions 5.7 and 5.9, concerns precisely the kind of change for which the marginal rate of substitution between a non-contingent change in first-period consumption and an equal non-contingent change in second-period consumption provides the appropriate welfare weights.

These welfare weights are akin to the different participants' assessments of rooms in Hilbert's infinite hotel, discussed in the introduction. No assets are involved, only preferences over consumption goods. If riskless assets are actually held, the implicit intertemporal relative price $R(\tilde{w}_t)$ coincides with the riskless rate of return r. In this case, as indicated by Proposition 4.1, the comparison of $R(\tilde{w}_t)$ with the growth rate turns into a comparison of r and the growth rate, but that is a coincidental finding.

The assessment of dynamic inefficiency hinges on the comparison of the marginal rate of substitution between non-contingent changes in first-period and second-period consumption to the growth rate because (i) the concept of constrained efficiency considers only interventions that do not alter the way in which generations t and $t+1$ share the risks attached to the productivity parameter $\tilde{A}_{t+1}$ and (ii) the population growth rate is certain. If as of period t the population growth rate between periods t and $t+1$ was uncertain, then with a non-contingent payment Δ_{t+1} of generation $t+1$ in period $t+1$, the transfers received by generation t in period $t+1$ must depend on the size of generation $t+1$, i.e. on the intervening population growth.

If there is a risky asset whose return pattern mimicks the risk pattern of the population growth rate, one might say that this is the asset to be considered in assessments of dynamic inefficiency. However, like the criterion of Proposition 4.1, such a singling out of a specific asset would be coincidental. The essence of the argument would turn on the appropriate intertemporal marginal rate of substitution, now between a non-contingent change in first-period consumption and a growth-contingent change in second-period consumption. Ultimately, the scope for Pareto-improving interventions, direct interventions or fiscal policies, does not depend on rates of return on assets but on comparisons of intertemporal marginal rates of substitution with the growth-determined terms of intertemporal trade that are offered by the peculiarities of the overlapping-generations model.

What about Crowding Out of Risky Investments? The assessment that dynamic ineffiency hinges only on the "riskless" intertemporal marginal rates of substitution $R(w), w \geq 0$, leaves open the question how we should think about the allocation of risky capital in this model. Could risky capital also be a source of dynamic inefficiency? The question is important if, for some reason, specific subsidies are not available to neutralize the effects on risky investments that a lump sum tax-and-transfer scheme might have.

In Blanchard (2019), a lump sum tax-and-transfer scheme without specific subsidies will crowd out private investments, causing price effects that benefit capitalists and harm workers. In the model of this paper, this conclusion is not generally true. There can be *crowding in as well as crowding out* of risky investment.

Proposition 6.10 *For any $\Delta > 0$, consider a fiscal policy $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ such that $\Delta^t = \Delta, \hat{\Delta}^t(w) \equiv \Delta$, and $\sigma^{t-1}(w) \equiv 0$ for all t, ad let $\tilde{c}_1^t = c_1^*(\tilde{w}_t, \Delta)$, $\tilde{k}_s^t = k_s^*(\tilde{w}_t, \Delta)$, $\tilde{k}_r^t = k_r^*(\tilde{w}_t, \Delta)$ be the associated equilibrium levels of first-period consumption, safe and risky real investments. If $r < 1$ and $k_s^*(\tilde{w}_t, 0) > 0$, then*

$$\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) \geq 0, \tag{6.16}$$

and the inequality is strict if $v(\cdot)$ exhibits strictly decreasing absolute risk aver-

sion. If $k_s^*(\tilde{w}_t, 0) = 0$, then, regardless of r and $R(\tilde{w}_t)$,

$$\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) < 0. \tag{6.17}$$

If $r < 1$ and $k_s^*(\tilde{w}_t, 0) > 0$, the combination of a first-period tax Δ and a second-period subsidy Δ is the equivalent of a net first-period subsidy $\frac{1-r}{r} \cdot \Delta$. This subsidy induces an increase in first-period consumption and, if risk aversion is strictly decreasing, an increase in risky investment. Both increases are accompanied by a decrease in safe investment. In contrast, if $k_s^*(\tilde{w}_t, 0) = 0$, the intervention always causes risky investment to go down. The reason is that the intervention reduces the need for a store of value; if they are the only store of value held, risky investments must go down.

Blanchard's result that a tax-and-transfer scheme by itself will crowd out risky investment is due to the assumption that there is only one real asset. As had already been stressed by Tobin (1963), with more than one asset, the comparative statics analysis of fiscal interventions must allow for changes in portfolio composition. In the present context, if $k_s^*(\tilde{w}_t, 0) > 0$, the fiscal intervention crowds out safe investments, for which it is a close substitute, but may crowd in risky investment if the income effects from the efficiency gain reduce risk aversion. By part (b) of Proposition 3.2, the condition $k_s^*(\tilde{w}_t, 0) > 0$ is always satisfied if $P(\{0\}) > 0$.

Price Effects Neither crowding out nor crowding in is *per se* good or bad. In the absence of externalities, any first-order effects of such adjustments on welfare must be zero because the parties choosing the variables in question have been optimizing and must be indifferent at the margin. In the present setting, this conclusion is true for adjustments in consumption and safe investments but *not* for adjustments in risky investments. Because labour supply is inelastic, reductions in risky investments cause subsequent wage rates to go down.[23]

Given the equation

$$\tilde{w}_{t+1} = \psi(\tilde{A}_{t+1}, \tilde{w}_t, \Delta) = \tilde{A}_{t+1} \cdot F_\ell(k_r^*(\tilde{w}_t, \Delta), \bar{L}) \tag{6.18}$$

for the wage rate at date $t + 1$, the marginal effect of the fiscal intervention on the wage rate is given as

$$\frac{\partial \psi}{\partial \Delta}(\tilde{A}_{t+1}, \tilde{w}_t, 0) = \tilde{A}_{t+1} \cdot F_{\ell k}(k_r^*(\tilde{w}_t, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0), \tag{6.19}$$

[23] If, instead of being inelastic, labour supply was perfectly elastic at a wage rate $\hat{w}$ corresponding to some constant marginal cost of labour, this pecuniary externality would be Pareto-irrelevant. Thus, in a model with land as a riskless asset, with production technologies that are additively separable in land, labour, and risky capital, with dynamic inefficiency due to ad-valorem transaction costs on land, Hellwig (2020/21) gives an example showing that fiscal interventions reducing the equilibrium value of land - and of transaction costs - can induce Pareto improvements even though they cause risky investments to go down. Whether the expected rate of return on risky investments is greater or smaller than the growth rate is irrelevant for the welfare assessment.

which is negative if $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) < 0$, the case of crowding out, and positive if $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) > 0$, the case of crowding in.

The counterpart of the change in the wage rate is a change in the rate of return on period t risky investments,

$$\tilde{A}_{t+1} \cdot \rho(\tilde{w}_t, \Delta) = \tilde{A}_{t+1} \cdot F_k(k_r^*(\tilde{w}_t, \Delta), \bar{L}), \tag{6.20}$$

with

$$\tilde{A}_{t+1} \cdot \frac{\partial \rho}{\partial \Delta}(\tilde{w}_t, 0) = \tilde{A}_{t+1} \cdot F_{kk}(k_r^*(\tilde{w}_t, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0), \tag{6.21}$$

which is positive if $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) < 0$, the case of crowding out, and negative if $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) > 0$.

If there is crowding out (crowding in) of risky investment in periods t and $t+1$, generation t loses (gains) from the effect of Δ on $\tilde{w}_{t+1}$ and gains (loses) from the effect on $\rho(\tilde{w}_{t+1}, \Delta)$. From an interim perspective, conditioning on the information available to this person, the overall impact of the price effects of a marginal increase in Δ, starting from $\Delta = 0$, on the person's expected utility is equal to

$$\begin{aligned} & u'(\tilde{c}_1^{t+1}) \cdot \tilde{A}_{t+1} \cdot \bar{L} \cdot F_{\ell k}(k_r^*(\tilde{w}_t, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) \\ & + E[v'(\tilde{c}_2^{t+1}) \cdot \tilde{A}_{t+2} | \tilde{w}_t] \cdot k_r^*(\tilde{w}_{t+1}, 0) \cdot F_{kk}(k_r^*(\tilde{w}_{t+1}, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t+1}, 0). \end{aligned} \tag{6.22}$$

Appendix B provides a detailed analysis of this expression. The main point to note is that the first term depends on the actual value of the random productivity parameter $\tilde{A}_{t+1}$ and that this term outweighs the second term if its value is very large and is outweighed by the second term if its value is close to zero. In an interim approach, generation $t+1$ knows the value of $\tilde{A}_{t+1}$ so, if the range of this random variable is large enough, there is no way to get an assessment of the price effects of the fiscal intervention that is independent of what this value is.

The dependence of (6.22) on $\tilde{A}_{t+1}$ reflects the fact that the change in $k_r^*(\tilde{w}_t, 0)$, the previous generation's risky investment, affects the risk exposure of generation $t+1$. Asking a member of generation $t+1$ whether he or she likes a policy intervention that causes crowding out, $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t'}, 0) < 0$, in all periods t', and to make this assessment on the basis of information available in period $t+1$ is a bit like asking a skier about accident insurance after the skiing vacation is over and the person asked knows whether there has been an accident or not.

To take account of the impact of price effects on risk allocation, one must take an *ex ante* perspective. Appendix B also goes into details on that. The problem here is that welfare assessments depend on the different participants' attitudes towards the risks associated with the random variable $\tilde{A}_{t+1}$. In the absence of *ex ante* arrangements for risk sharing between the members of the different generations, there is no reason why the *laissez-faire* allocation of these risks should be efficient, nor is there any presumption as to the direction in which a reallocation of risks would provide for an improvement.

We thus face the deeper question whether and how policy should attempt to deal with the inefficiency of the risk allocation that is due to the absence of markets for intergenerational risk sharing. This question however seems unrelated to the issue of dynamic inefficiency.

Payroll Taxes. The fiscal interventions in Propositions 5.7 and 5.9 rely on lump sum taxes and subsidies. Lump sum taxes are a theorist's dream but not a real-world fiscal instrument. The question is whether they might be replaced by other policy instruments.

In a model of overlapping generations, with people saving for retirement, it is natural to ask whether the lump sum taxes can be replaced by the kind of payroll taxes that are commonly used in pay-as-you-go retirement systems. Payroll taxes introduce two complications into the analysis. First, since aggregate wage bills depend on the realizations of aggregate uncertainty, with constant tax rates, revenues from payroll taxes also depend on aggregate uncertainty. Second, if labour supplies are elastic, payroll taxes affect work incentives.

The first complication disappears if the payroll tax rate in each period is made to depend on the prevailing wage rate so that, for a given target $\Delta > 0$ for tax revenue, if the prevailing wage rate is $w \geq 0$, the payroll tax rate is

$$\tau_\Delta(w) := \frac{\Delta}{w \cdot \bar{L}}. \tag{6.23}$$

With this specification of payroll taxes, the conclusions of Propositions 5.7 and 5.9 remain valid.

In the real world of course payroll taxes are not adjusted to the state of the economy. Rather, the tax rates are by and large fixed, and fluctuations in tax revenues are smoothed through reserve management. This practice is at odds with the assumption that Δ is independent of the state of the economy. This assumption, however, was only introduced to separate the analysis of dynamic inefficiency from the analysis of inefficient risk allocations in an incomplete market system. The question is whether a smoothing of payroll taxation that might be provided by allowing Δ to depend on the state of the economy would improve overall risk allocation. This question needs to be studied in a more systematic analysis of policy challenges from aggregate risks.

The second complication, concerning incentive effects of payroll taxes, is moot if labour supplies are perfectly inelastic, as assumed in this paper. If labour supplies are elastic, e.g., if the utility specification (2.7) is replaced by the specification

$$u(c_1^t) + v(c_2^t) - \Gamma(L^t) \tag{6.24}$$

for some increasing convex function Γ of the person's labour supply L^t, the impact of payroll taxes depends on whether benefits from the system are proportional to contributions or not.[24] If benefits are independent of contributions,

[24] See Homburg (1990), Breyer and Straub (1993), Hellwig (2020/2021).

payroll taxes induce a downward distortion into labour supplies. The appropriate analogues of Propositions 5.7 and 5.9 will still hold because these propositions concern only the first-order effects from a small fiscal intervention, starting from the *laissez-faire* allocation, and as is well known the initial distortionary effects of a specific tax are zero. The distortions would matter, however, for an analysis of optimal fiscal interventions, which I have not engaged in here.

If benefits are equal to contributions, the downward distortions of labour supplies from the tax part of the system must be balanced against the upward distortion from the benefits part of the system.[25] In the present context, suppose that, at the margin, any increase in contributions raises benefits by the same amount. Then, in the maximization of expected utility, the first-order condition for labour supply in period t takes the form

$$\Gamma'(\tilde{L}^t) = (1 - \tau_\Delta(\tilde{w}_t)) \cdot \tilde{w}_t \cdot u'(\tilde{c}_1^t) + \tau_\Delta(\tilde{w}_t) \cdot \tilde{w}_t \cdot Ev'(\tilde{c}_2^t), \tag{6.25}$$

which by (4.5) can be rewritten as

$$\Gamma'(\tilde{L}^t) = \left(1 + \tau_\Delta(\tilde{w}_t)\left(\frac{1}{R(\tilde{w}_t)} - 1\right)\right) \cdot \tilde{w}_t \cdot u'(\tilde{c}_1^t). \tag{6.26}$$

Thus in the constellation of Proposition 5.7, with $R(w) < 1$ for all $w \geq 0$, the incentive effects from benefits outweigh the incentive effects from payroll taxes. The reduction of dynamic inefficiency also improves labour incentives.[26]

Public Debt. Another alternative to the lump sum taxes and subsidies in Propositions 5.7 and 5.9 is public debt. Blanchard's (2019) analysis is part of a lecture on "Public Debt and Low Interest Rates", accompanied by the observation that, "under certainty and in steady state", the lump sum tax-and-subsidy scheme he considers is equivalent to a scheme for issuing and rolling over public debt. What happens to this equivalence when there is uncertainty and the notion of steady state applies only at the level of probability distributions, not at the level of realizations?

Consider Proposition 5.7. Suppose that in period t the amount $\Delta > 0$ is to be raised from the young by issuing debt rather than by a lump sum tax. In contrast to the lump sum tax, the debt issue cannot be sold by government fiat but must be acceptable to the buyers. Assuming that Δ is small, acceptability can be taken for granted if the debt promises a rate of return greater than $R(\tilde{w}_t)$, the *laissez-faire* equilibrium value of the implicit relative price of consumption in period t versus consumption in period $t+1$. If $R(\tilde{w}_t) < 1$ almost surely, as assumed in Poposition 5.7, such a promise is certainly feasible because the payment owed can be taken to be no larger than the amount Δ that will be raised by issuing new debt to the next generation. If the government promises to repay Δ, there actually will be an excess demand for this debt in

[25] This was the point of the reaction of Breyer and Straub (1993) to Homburg (1990).

[26] For a full welfare analysis, albeit in a somewhat different model, see Hellwig (2020/2021).

period t. Alternatively, if the government were to offer any amount of debt with a promised rate of return equal to one, government debt would completely crowd out safe investments (if any are made under *laissez faire*), with a very substantial, non-marginal change in the equilibrium allocation.

If the government promises to repay an amount between Δ and $R(\tilde{w}_t)\cdot\Delta$, the excess of the next period's revenue from issuing old debt over the amount needed to repay new debt can be used to finance other things, e.g., the subsidies to risky investments needed to neutralize incentive effects. To the extent that such subsidies (or taxes) are used, some lump sum element of government funding (or spending) may be needed even when resources from the young in each period are obtained by issuing public debt.

What about Proposition 5.9? If $R(\tilde{w}_t) < 1$ almost surely, the preceding argument goes through unchanged. However, the condition for an *ex ante* Pareto-improving fiscal intervention in Proposition 5.9 requires only that a marginal-utility weighted mean of $R(\tilde{w}_t)$ be less than one, so there may be periods and states of the economy where $R(\tilde{w}_t) > 1$ even though the condition for an *ex ante* beneficial fiscal intervention is satisfied. In such periods and states of the economy, the payment promises needed to get the young to accept the new debt issue exceed the amount of the next period's debt issue. Then it is not clear that, if the condition for dynamic inefficiency of the *laissez-faire* allocation is satisfied, a Pareto improvement can be implemented by a fiscal intervention based on issuing and rolling over public debt. Here again we encounter the issue of how to think about fiscal interventions that condition on the state and the history of the economy.

Fiat Money. In many deterministic models, a scheme of issuing and rolling over public debt with a maturity of one period is equivalent to an arrangement with what Samuelson (1958) called the *social contrivance of money*, a paper asset, that is issued once and then circulates forever.[27] Thus in the simple consumption loan model, a Pareto efficient allocation can be implemented by having old people in period 1 use pieces of paper with printed pictures of George Washington to buy the good of period 1 from people of generation 1, who in turn will use the pieces of paper to buy the good of period 2 from the people of generation 2, and so on. Similarly, in the infinite hotel, guest no. 1 might use such a piece of paper to buy the right to occupy room 2 from guest no. 2, who in turn uses it to buy the right to occupy room 3 from guest no. 3, and so on.

In the stochastic model of this paper, this equivalence does not hold. Whereas the tax-and-transfer schemes analysed above, and their public-debt analogues, provide for non-contingent changes in the allocation of consumption goods, the allocative effects of using fiat money would have to be state-contingent: In any period t, the value w_t of the wage rate random variable $\tilde{w}_t$ determines the budget of people born in period t and therefore the resources they (can) devote to

[27] For a systematic analysis of the relation between dynamic inefficiency of laissez-faire allocations and the existence of equilibria with paper assets, see Tirole (1985).

acquiring money as a store of value. If w_t is small, their demand for money must be small, if w_t is large, their demand for money can be large.

This state dependence of the demand for money translates into a state dependence of the equilibrium real value of money: If w_t is small, the equilibrium real value of the given nominal quantity of fiat money in period t must be small, if w_t is large, the equilibrium real value of money can be large.

This state dependence of the equilibrium real value of fiat money also implies that the rate of return to holding money is subject to uncertainty. People deciding to hold money from period t to period $t+1$ must take account of the fact that the purchasing power of this store of value in period $t+1$ depends on the resources available in that period. As an asset, fiat money is risky rather than safe, and its returns are correlated with the returns on risky capital. If the utility functions u and v are logarithmic, the rate of return on fiat money will in fact be a convex combination of a safe rate of return and the rate of return on risky capital. Reliance on fiat money as a store of value facilitating intergenerational exchange is likely to crowd out investments in risky capital.

The existence and welfare properties of a monetary equilibrium, including a comparison to the *laissez-faire* equilibrium without fiat money, need further investigation. A natural question to ask is what is the scope and what are the welfare effects of a monetary-cum-fiscal policy that would smooth fluctuations in the equilibrium real value of money.

Are We in a Situation of Dynamic Inefficiency? In many OECD countries, developments of the past four decades have led to a situation where, for some time now, rates of return on safe assets have been below growth rates and mean rates of return on risky assets (all assets) have been above growth rates. The notion that safe assets are "scarce" has become a major theme in macroeconomics, in particular international monetary macroeconomics.[28] This is precisely the constellation where an assessment of dynamic inefficiency depends on which rates of return one deems to be relevant.

In the tradition of Abel et al. (1989), many authors have taken for granted that assessments of dynamic inefficiency must involve rates of return on all assets and therefore, that we are *not* in a situation of dynamic inefficiency.[29] The present paper shows that this assessment is unwarranted. An assessment of dynamic inefficiency should be based on the safe rate of return.[30]

I would however be cautious about the conclusion that we actually are in a situation of dynamic inefficiency. Such a conclusion would require a translation

[28] Caballero et al. (2017).

[29] See, e.g., Homburg (2014), Acharya and Droga (2020), Reis (2020).

[30] This position is also taken by von Weizsäcker and Krämer (2019). They argue that appropriate measures of rates of return on risky assets are also below growth rates when one deducts the relevant risk premia. Whereas the discussion here focuses on factors underlying the growth of the demand for safe assets, von Weizsäcker and Krämer, like von Weizsäcker (2014) and Rachel and Summers (2019), treat the observed declines in rates of return as a consequence of growth in the overall demand for stores of value.

of the model studied in this paper into a real-world application. Such a translation must overcome two obstacles. First, the model involves equilibria that follow time homogeneous Markov processes. Stationarity cannot be presumed in the real world, so the question is how to assess the possibility that the current constellation of safe interest rates and growth rates might not last.

Second, the model has no institutions, no money, no banks, and no money market funds engaging in liquidity, maturity, and risk transformation. Monetary and financial arrangements and institutions have however played a significant role in the developments that have led to the current situation. Increases in the demand for safe assets over the past few decades had a lot to do with emerging economies' reacting to the Asian crisis of the late 1990s by building up reserves of safe dollar assets, pushing money market mutual funds out of Treasuries and into secured lending to banks. The dramatic expansion of liquidity creation by banks and money market funds in the runup to the financial crisis of 2007-2009 involved ever more "safe" assets as collateral. Some of the "safeness", however, was merely an illusion that collapsed in the crisis.[31] The subsequent shift of many banks from a reliance on markets to a reliance on reserves for liquidity added to the demand for central bank money as the ultimate safe asset.

To argue that we are in a situation of dynamic inefficiency, one would have to show that the forces underlying the developments of the past two decades bear some relation to the forces at work in the theoretical analysis, a high demand for a store of value and a high demand for safe returns, implying that large risk premia must be deducted from expected rates of return on risky assets. By its very nature, such an identification exercise cannot do with the abstract model considered here but requires a framework that allows for monetary and financial institutions of the sort that have played a role in the developments at issue.[32]

A Proofs

A.1 Proofs for Section 3

Before turning to the proof of Proposition 3.1 as such, for any $w \geq 0$ and any $\rho > 0$, I consider the problem of choosing c_1, k_s, k_r to maximize

$$u(c_1) + \int v\left(r \cdot k_s + A \cdot \rho \cdot k_r\right) dP(A) \tag{A.1}$$

[31] See Admati and Hellwig (2013), Ch. 10.

[32] Another issue concerns the role of fiat money. As was discussed above, the very existence of fiat money can be seen as a reflection of an underlying problem of dynamic inefficiency. However, in many overlapping-generations models with fiat money and sequentially complete markets, dynamic inefficiency cannot arise because any need for a store of value is met by a revaluation of fiat money. This observation raises the question whether we should think about the presumed scarcity of safe assets in the real world as an equilibrium phenomenon, as in the theory of dynamic inefficiency, or as a disequilibrium phenomenon, due to an inability of money prices to fall frictionlessly to raise the real value of the available quantity of fiat money.

subject to the constraint

$$c_1 + k_s + k_r = e + w\bar{L}. \tag{A.2}$$

Recall that $A^* = \int AdP(A)$ is the mean of the productivity parameter under the distribution P.

Lemma A.1 *For any $w \geq 0$ and $\rho > 0$, the problem of maximizing (A.1) under the constraint (A.2) has a unique solution $(c_1(w,\rho), k_s(w,\rho), k_r(w,\rho))$. The solution depends continuously on w and ρ. With v exhibiting positive, nonincreasing absolute risk aversion, the functions $c_1(\cdot,\cdot)$, $k_s(\cdot,\cdot)$ and $k_r(\cdot,\cdot)$ have the following properties:*

(a) For any $\rho > 0$, $c_1(w,\rho)$ is increasing in w and bounded away from zero;
(b) For any $w \geq 0$, the following are true:
(b.1) If $A^* \cdot \rho < r$, then $k_s(w,\rho) > 0$ and $k_r(w,\rho) = 0$.
(b.2) If $P(\{0\}) > 0$, then $k_s(w,\rho) > 0$. If $P([0, \frac{r}{\rho})) = 0$, then $k_s(w,\rho) = 0$.
(b.3) If $A^* \cdot \rho > r$, then $k_r(w,\rho) > 0$; moreover,

$$\frac{\partial k_r}{\partial w}(w,\rho) \geq 0 \quad \textit{and} \quad \frac{\partial k_r}{\partial \rho}(w,\rho) > -\frac{k_r}{\rho}. \tag{A.3}$$

(b.4) $\lim_{\rho\to\infty}(\rho \cdot k_r(w,\rho)) = \infty$.

Proof. For any $w \geq 0$ and $\rho > 0$, the triple (c_1, k_s, k_r) maximizes (A.1) under the constraint (A.2) if and only if, for $q = \frac{1}{\rho}$ and $B = e + w\bar{L}$, c_1, k_s and $I = \rho k_r$ maximize the expression

$$u(c_1) + \int v\left(r \cdot k_s + I(A - qr)\right)dP(A) \tag{A.4}$$

under the constraint

$$c_1 + k_s + qI = B. \tag{A.5}$$

Existence and uniqueness of a solution to this problem follow from the continuity and the strict concavity of $u(\cdot)$ and $v(\cdot)$. Continuity of the solution follows from the maximum theorem. The first two statements of the lemma follow immediately. The third statement is equivalent to the statement that the solution to the problem of maximizing (A.4) satisfies:

- (a*) c_1 is increasing in B;
- (b.1*) for $P(\{0\}) > 0$, $k_s > 0$; for $P([0, qr)) = 0$, $k_s = 0$;
- (b.2*) for $qr \geq A^*$, $I = 0$; for $qr < A^*$, $I > 0$
- (b.3*) if $qr < A^*$, then $\frac{\partial I}{\partial B} \geq 0$ and $\frac{\partial I}{\partial q} < 0$;
- (b.4*) $\lim_{q\to 0} I = \infty$.

To prove these statements, I consider the first-order conditions for the maximization of (A.4) subject to (A.5):

$$u'(c_1) - \lambda = 0, \tag{A.6}$$

$$r \int v'(r \cdot k_s + A \cdot I)\, dP(A) - \lambda \leq 0, \tag{A.7}$$

$$\int v'(r \cdot k_s + A \cdot I) \cdot A\, dP(A) - q\lambda \leq 0, \tag{A.8}$$

where λ is a Lagrange multiplier for the constraint (A.5), and, if one of the inequalities (A.7), (A.8) is strict, the corresponding variable in the maximization takes the value zero. Since $u'(0) = \infty$, a boundary solution for c_1 is ruled out. Similarly, since $v'(0) = \infty$, a boundary solution for both k_s and I is also ruled out.

Since $v'(0) = \infty$, $P(\{0\}) > 0$ must imply $k_s > 0$ since otherwise the left-hand side of (A.7) would be unbounded. If $P([0, qr)) = 0$, $A \geq qr$, P-almost surely, and $A > qr$ with positive P-probability. In this case, (A.8) implies that (A.7) holds with a strict inequality and, hence, that $k_s = 0$. Statement (b.1*) is thus proved.

Turning to statement (b.2*), I note that, if $I > 0$, (A.8) must hold as an equation and, moreover, the left-hand side of (A.8) is less than $\int v'\,(r \cdot k_s + I(A - qr))dP(A) \cdot A^*$. Hence $\int v'\,(r \cdot k_s + I(A - qr))dP(A) \cdot A^* > q\lambda$. By (A.7) therefore, $A^* > qr$. Conversely, $A^* \leq qr$ implies $I = 0$. If $I = 0$, (A.7) and (A.8) take the form $rv'(rk_s) - \lambda = 0$ and $v'(rk_s) \cdot A^* - q\lambda \leq 0$, implying that $A^* \leq qr$. Conversely, $A^* > qr$ implies $I > 0$. Statement (b.2*) is thus proved.

For statement (a*) and (b.3*), I distinguish two cases, according to whether $k_s > 0$ or $k_s = 0$. If $qr < A^*$ and $k_s > 0$, conditions (A.7) and (A.8) hold as equations. Substitution from (A.7) in (A.6) and (A.8) yields

$$u'(c_1) - r \int v'(r \cdot k_s + A \cdot I)\, dP(A) = 0 \tag{A.9}$$

and

$$\int v'(r \cdot k_s + A \cdot I) \cdot (A - qr)\, dP(A) = 0, \tag{A.10}$$

with a strict inequality only if $I = 0$. By the implicit function theorem, (A.9) and (A.10) imply:

$$\begin{pmatrix} u'' + r^2\, Ev'' & -rEv''(\tilde{A} - qr) \\ -rEv''(\tilde{A} - qr) & Ev''(\tilde{A} - qr)^2 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial c_1}{\partial B} \\ \frac{\partial I}{\partial B} \end{pmatrix} = \begin{pmatrix} r^2 Ev'' \\ rEv''(\tilde{A} - qr) \end{pmatrix} \tag{A.11}$$

$$\begin{pmatrix} u'' + r^2\, Ev'' & -rEv''(\tilde{A} - qr) \\ -rEv''(\tilde{A} - qr) & Ev''(\tilde{A} - qr)^2 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial c_1}{\partial q} \\ \frac{\partial I}{\partial q} \end{pmatrix} = \begin{pmatrix} r^2 Ev'' I \\ -rEv''(\tilde{A} - qr)I - rEv' \end{pmatrix}, \tag{A.12}$$

where I have dropped the arguments of the derivatives u'', v'', and v'. From (A.11) and (A.12), one obtains

$$\frac{\partial c_1}{\partial B} = \frac{|H_v|}{|H|} \tag{A.13}$$

$$\frac{\partial I}{\partial B} = \frac{1}{|H|} \cdot [(u'' + r^2\ Ev'') \cdot rEv''(\tilde{A} - qr) + rEv''(\tilde{A} - qr) \cdot r^2 Ev'' \quad (A.14)$$

and

$$\frac{\partial I}{\partial q} \quad = \quad \frac{1}{|H|} \cdot [(u'' + r^2\ Ev'') \cdot (rEv''(\tilde{A} - qr)I + rEv') + rEv''(\tilde{A} - qr) \cdot r^2 Ev'' I], \tag{A.15}$$

where $|H|$ is the determinant of the two-by-two matrix on the left-hand side of (A.11) and (A.12), the Hessian of the function

$$(c_1, I) \longmapsto u(c_1) + v\,(r \cdot (B - c_1) + I(A - qr)) \tag{A.16}$$

and $|H_v|$ is the determinant of the Hessian of the function

$$(c_1, I) \longmapsto v\,(r \cdot (B - c_1) + I(A - qr)) \tag{A.17}$$

Because the functions (A.16) and (A.17) are strictly concave, the determinants $|H|$ and $|H_v|$ are strictly positive. Thus, $\frac{\partial c_1}{\partial B} > 0$, which proves (a*) for the case $k_s > 0$

Also, by standard arguments,[33] nonincreasing absolute risk aversion and condition (A.7) imply that

$$Ev'' r(\tilde{A} - qr) \geq 0, \tag{A.18}$$

and the inequality is strict if risk aversion is strictly decreasing. Since $u'' < 0$ and $v'' < 0$, it follows that (A.14) and (A.15) imply

$$\frac{\partial I}{\partial B} \geq 0 \tag{A.19}$$

and

$$\frac{\partial I}{\partial q} < 0, \tag{A.20}$$

which proves (b.3*) for the case $k_s > 0$.

If $k_s = 0$, $k_r > 0$, $qr < A^*$, (A.6) and (A.8) yield

$$qu'(c_1) - \int v'(A \cdot I) \cdot A\ dP(A) = 0. \tag{A.21}$$

Again using the implicit function theorem, one obtains

$$\frac{\partial c_1}{\partial B} = \frac{\int v'(A \cdot I) \cdot A\ dP(A)}{q^2 u''(c_1) + \int v''(A \cdot I) \cdot A\ dP(A)} \in (0, 1),$$

[33] See, e.g. LeRoy and Werner (2001), p. 119.

$$\frac{\partial I}{\partial B} = \frac{q^2 u''(c_1)}{q^2 u''(c_1) + \int v''(A \cdot I) \cdot A \, dP(A)} \in (0,1),$$

and

$$\frac{\partial I}{\partial q} = \frac{u'(c_1)}{q^2 u''(c_1) + \int v''(A \cdot I) \cdot A \, dP(A)} < 0,$$

which proves (a*) and (b.3*) for the case $k_s = 0$.

Finally, to prove (b.4*), I note that $\int v'(r \cdot k_s + A \cdot I) \cdot A \, dP(A) \geq v'(r \cdot B + I \cdot \bar{A}) \cdot A^*$, so (A.6) and (A.8) imply

$$v'(r \cdot B + I \cdot \bar{A}) \cdot A^* \leq q u'(c_1). \tag{A.22}$$

If I were bounded, uniformly in q, the left-hand side of (A.21) would be bounded away from zero as q goes to zero. For very small q therefore, $u'(c_1)$ must be very large. Then there exists $\varepsilon > 0$ such that, for $q < \varepsilon$, $c_1 + Iq < \frac{B}{2}$ and $k_s > \frac{B}{2}$. For such q, $r \cdot k_s + \tilde{A} \cdot I > r \cdot \frac{B}{2}$ almost surely, so (A.7) and (A.8) imply

$$v'(r \cdot B + I \cdot \bar{A}) \cdot A^* \leq q r v'\left(r \cdot \frac{B}{2}\right),$$

which is impossible if q is close to zero and I is bounded. The assumption that I is bounded, uniformly in q, thus leads to a contradiction and must be false. Statement (b.4*) follows immediately. ■

Lemma A.2 *For any $w \geq 0$, there exists a unique $\bar{k}(w) > 0$ such that the function $k_r(\cdot, \cdot)$ in Lemma A.1 satisfies*

$$k_r(w, F_k(\bar{k}(w), \bar{L})) = \bar{k}(w). \tag{A.23}$$

Proof. Consider the functions $\bar{k} \longmapsto F_k(\bar{k}, \bar{L}) \cdot k_r(w, F_k(\bar{k}, \bar{L}))$ and $\bar{k} \longmapsto F_k(\bar{k}, \bar{L}) \cdot \bar{k}$. For $\bar{k} > e + w\bar{L}$, trivially, $k_r(w, F_k(\bar{k}, \bar{L})) < \bar{k}$ and therefore also

$$F_k(\bar{k}, \bar{L}) \cdot k_r(w, F_k(\bar{k}, \bar{L})) < F_k(\bar{k}, \bar{L}) \cdot \bar{k}.$$

By assumption, the function $\bar{k} \longmapsto F_k(\bar{k}, \bar{L}) \cdot \bar{k}$ is increasing so, for $\bar{k} \leq e + w\bar{L}$, $F_k(\bar{k}, \bar{L}) \cdot \bar{k}$ is bounded. Since $\lim_{\bar{k} \to 0} F_k(\bar{k}, \bar{L}) = \infty$, statement (c) in Lemma A.1 implies that

$$\lim_{\bar{k} \to 0} [F_k(\bar{k}, \bar{L}) \cdot k_r(w, F_k(\bar{k}, \bar{L}))] = \infty,$$

and therefore

$$F_k(\bar{k}, \bar{L}) \cdot k_r(w, F_k(\bar{k}, \bar{L})) > F_k(\bar{k}, \bar{L}) \cdot \bar{k}$$

for $\bar{k}$ sufficiently close to zero. By the intermediate value theorem, it follows that there exists $\bar{k} \in (0, e + w\bar{L})$ such that

$$F_k(\bar{k}, \bar{L}) \cdot k_r(w, F_k(\bar{k}, \bar{L})) = F_k(\bar{k}, \bar{L}) \cdot \bar{k} \tag{A.24}$$

and therefore

$$k_r(w, F_k(\bar{k}, \bar{L})) = \bar{k}.$$

Uniqueness of the solution to (A.24) follows because, by statement (b) in Lemma A.1 and the strict concavity of the production function, the left-hand side of (A.24) is decreasing and, by assumption, the right-hand side of (A.24) is increasing in $\bar{k}$. ■

Proof of Proposition 3.1. For φ and ψ defined by statements (i) and (ii), the validity of (3.1) and (3.2) follows from the first-order condition (3.11) and the labour market clearing condition (2.15). For λ given by statement (iii), equation (3.4) also follows from the first-order condition (3.11) in the text, together with the constant-returns-to-scale property of F. For ρ given by statement (iv), (3.5) follows from the argument in the text that leads from equation (3.12) to equation (3.15).

By the principle of dynamic programming, for any t, the triple $(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t)$ must maximize

$$E[u(\tilde{c}_1^t) + v(r\tilde{k}_s^t + \tilde{A}_{t+1}\rho(\tilde{w}_t)\tilde{k}_r^t)] \tag{A.25}$$

under the constraint

$$\tilde{c}_1^t + \tilde{k}_s^t + \tilde{k}_r^t = e + \tilde{w}_t\bar{L}. \tag{A.26}$$

For the given specifications of $\tilde{w}_t$ and $\rho(\tilde{w}_t)$, therefore, $(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t)$ takes the form (3.3) where for any $w \geq 0$, $(c_1^*(w), k_s^*(w), k_r^*(w))$ maximizes

$$u(c_1) + Ev(rk_s + \tilde{A}\rho(w)k_r)$$

under the constraint

$$c_1 + k_s + k_r = e + w\bar{L}.$$

Statement (v) follows immediately.

For any $w \geq 0$, therefore, it follows that the triple $(c_1^*(w), k_s^*(w), k_r^*(w))$ is equal to the solution $(c_1(w, \rho(w)), k_s(w, \rho(w)), k_r(w, \rho(w)))$ to the problem of maximizing (A.1) under the constraint (A.2) when the wage rate is w and risky-return parameter is $\rho(w)$. By (3.8), therefore,

$$k_r^*(w) = k_r(w, \rho(w)) = k_r(w, F_k(k_r^*(w), \bar{L}).$$

By Lemma A.2, this is only possible if $k_r^*(w) = \bar{k}(w)$ and $\rho(w) = F_k(\bar{k}(w), \bar{L})$, where $\bar{k}(w)$ is the unique solution to (A.23). Thus, for any $w \geq 0$,

$$(c_1^*(w), k_s^*(w), k_r^*(w)) = (c_1(w, F_k(\bar{k}(w), \bar{L})), k_s(w, F_k(\bar{k}(w), \bar{L})), k_r(w, F_k(\bar{k}(w), \bar{L}))). \tag{A.27}$$

Uniqueness of equilibrium follows from the uniqueness of the solutions to (A.23) and to the problem of maximizing (A.1) under the constraint (A.2).

One easily checks that the specified conditions are sufficient as well as necessary for an equilibrium. This completes the proof of Proposition 3.1. ■

Proof of Proposition 3.2. I begin with the proof of statement (c). I first show that the function $w \longmapsto k_r^*(w)$ is nondecreasing. By construction, this claim is equivalent to the claim that the function $w \longmapsto \bar{k}(w)$ that is defined by

Lemma A.2 is nondecreasing. Suppose that the claim is false and that $\frac{d\bar{k}}{dw} < 0$ for some $w \geq 0$.

From (A.23), one has

$$\frac{d\bar{k}}{dw} = \frac{\partial k_r}{\partial w}(w, F_k(\bar{k}(w), \bar{L}) + \frac{\partial k_r}{\partial \rho}(w, F_k(\bar{k}(w), \bar{L}) \cdot F_{kk}(\bar{k}(w), \bar{L}) \cdot \frac{d\bar{k}}{dw}. \quad \text{(A.28)}$$

By statement (b.3) in Lemma A.1, the first term on the right-hand side of (A.28) is nonnegative. Hence,

$$\frac{d\bar{k}}{dw} \cdot \left(1 + \frac{k_r(w, F_k(\bar{k}(w), \bar{L})}{F_k(\bar{k}(w), \bar{L})} \cdot F_{kk}(\bar{k}(w), \bar{L})\right) \geq 0. \quad \text{(A.29)}$$

If $\frac{d\bar{k}}{dw}$ were negative, the product, $F_{kk}(\bar{k}(w), \bar{L}) \cdot \frac{d\bar{k}}{dw}$ would have to be positive, so by statement (b.3) in Lemma A.1, (A.29) would imply

$$1 + \frac{k_r(w, F_k(\bar{k}(w), \bar{L})}{F_k(\bar{k}(w), \bar{L})} \cdot F_{kk}(\bar{k}(w), \bar{L}) \leq 0,$$

which is incompatible with the assumption that the function $\bar{k} \longmapsto F_k(\bar{k}, \bar{L}) \cdot k_r(w, F_k(\bar{k}, \bar{L}))$ is increasing. The assumption that $\frac{d\bar{k}}{dw} < 0$ for some $w \geq 0$ thus leads to a contradiction and must be false.

Because $k_r^*(w) = \bar{k}(w)$ is nondecreasing in w, a lower bound $\underline{K}$ for $\bar{k}(w)$ is obtained by setting $\underline{K} = \bar{k}(e)$. By Lemma A.2, $\underline{K} > 0$. As for an upper bound, let $\bar{K}$ be such that

$$\int A dP(A) \cdot F_k(\bar{K}, \bar{L}) < r. \quad \text{(A.30)}$$

By statement (b.1) in Lemma A.1, (A.30) implies that $k_r(w, F_k(\bar{K}, \bar{L})) = 0 < \bar{k}(w)$ for all w. Hence $\bar{k}(w) < \bar{K}$ for all w. This completes the proof of statement (c) in Proposition 3.2.

Statement (b) follows from statement (b.2) in Lemma A.1, statement (d) from statement (b.1) in Lemma A.1 and the positivity of $k_r^*(w)$.

As for statement (a), I note that, for all $w \geq 0$, $c_1^*(w) = c_1(w, \rho(w))$ where $\rho(w) = F_k(\bar{k}(w), \bar{L})$. By statement (a) in Lemma A.1, it follows that, for all $w \geq 0$, $c_1^*(w) \geq c_1(e, \rho(w))$. By statement (c), therefore, for all $w \geq 0$, $c_1^*(w) \geq \inf_{k \in [\underline{K}, \bar{K}]} c_1(0, F_k(k, \bar{L}))$. By the continuity of $c_1(\cdot, \cdot)$, the infimum is actually a minimum and is attained at some $\hat{k} \in [\underline{K}, \bar{K}]$. Upon setting $\underline{c} = c_1(0, F_k(\hat{k}, \bar{L}))$, one obtains statement (a). ■

Proposition 3.3 follows from Corollary 4, p. 1392, and Theorem 2, p. 1397, of Hopenhayn and Prescott (1992).

A.2 Proofs for Section 4

Proof of Proposition 4.1. The proof of the first statement is indirect. Suppose that the statement is false, that $\tilde{k}_s^t > 0$ and $r > 1$, and that, for some sequence $\{\Delta_t\}_{t=1}^{\infty}$ the allocation $(\tilde{\ell}_r^0, \tilde{c}_2^0 + \Delta_1), \{(\tilde{c}_1^t - \Delta_t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t + \Delta_{t+1})\}_{t=1}^{\infty}$ is *interim* Pareto-preferred to the equilibrium allocation $(\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$. Then $\Delta_1 \geq 0$ and, for $t \geq 1$,

$$u(c_1^*(\tilde{w}_t) - \Delta_t) + \int v(rk_s^*(\tilde{w}_t) + A\rho(\tilde{w}_t)k_r^*(\tilde{w}_t) + \Delta_{t+1})\, dP(A)$$
$$\geq u(c_1^*(\tilde{w}_t)) + \int v(rk_s^*(\tilde{w}_t) + A\rho(\tilde{w}_t)k_r^*(\tilde{w}_t))\, dP(A), \qquad \text{(A.31)}$$

and the inequality is strict for some t. Since $\Delta_1 \geq 0$, a straight forward induction implies that $\Delta_t \geq 0$ for all t; moreover, if $\Delta_t > 0$, then $\Delta_{t'} > 0$ for all $t' > t$. Let t^* be the first t for which $\Delta_t > 0$. (Existence of t^* is implied by the fact that the inequality in (A.31) is strict for some t.) By the strict concavity of $u(\cdot)$ and $v(\cdot)$, for $t \geq t^*$, (A.31) implies that

$$-\Delta_t \cdot u \cdot (c_1^*(\tilde{w}_t)) + \Delta_{t+1} \cdot \int v'(rk_s^*(\tilde{w}_t) + A\rho(\tilde{w}_t)k_r^*(\tilde{w}_t))\, dP(A) > 0. \quad \text{(A.32)}$$

By the first-order condition (3.16) for the choice of $c_1^*(\tilde{w}_t) > 0$ and $k_s^*(\tilde{w}_t) > 0$, (A.32) is equivalent to the inequality

$$-\Delta_t \cdot u'(c_1^*(\tilde{w}_t)) + \frac{1}{r} \cdot \Delta_{t+1} \cdot u'(c_1^*(\tilde{w}_t)) > 0, \qquad \text{(A.33)}$$

which in turn is equivalent to the inequality

$$\Delta_t < \frac{1}{r} \cdot \Delta_{t+1}. \qquad \text{(A.34)}$$

Since (A.34) must hold for all $t \geq t^*$, it follows that

$$\Delta_{t^*} < \frac{1}{r^s} \cdot \Delta_{t^*+s} \qquad \text{(A.35)}$$

for all s. Feasibility requires that $\Delta_{t^*+s} \leq c_1^*(\tilde{w}_{t^*+s})$ for all s and therefore $\Delta_{t^*+s} \leq e + \tilde{w}_{t^*+s}\bar{L} \leq e + \bar{A}F_\ell(\bar{K}, \bar{L})$, where $\bar{K}$ is the bound given by Proposition 3.2. Since Δ_{t^*+s} is uniformaly bounded, the validity of (A.35) for all s implies that $\Delta_{t^*} = 0$. The assumption that the first statement of the proposition is false has thus led to a contradiction, which proves the statement. The second statement follows by the argument given in the text. ■

Corollary 4.2 follows from Propositions 4.1 and 3.2.

Proof of Proposition 4.3. The proof of the first statement is again indirect. Suppose that the statement is false, that $G^*(\{w \geq 0 | R(w) > 1\}) > 0$, and that, for some sequence $\{\Delta_t\}_{t=1}^{\infty}$ the allocation $(\tilde{\ell}_r^0, \tilde{c}_2^0 + \Delta_1), \{(\tilde{c}_1^t - \Delta_t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t + \Delta_{t+1})\}_{t=1}^{\infty}$ is *interim* Pareto-preferred to the equilibrium allocation $(\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$. Then, by the same argument as in the proof of Proposition 4.1, $\Delta_t \geq 0$ for all t; moreover, if $\Delta_t > 0$, then $\Delta_{t'} > 0$ for all $t' > t$. Let t^* be the first t for which $\Delta_t > 0$. (Existence of t^* is implied by the fact that the inequality in (A.31) is strict for some t.) By the strict concavity of $u(\cdot)$ and $v(\cdot)$, for $t \geq t^*$ and $\tilde{w}_t = w$, (A.32) takes the form

$$-\Delta_t \cdot u \cdot (c_1^*(w)) + \Delta_{t+1} \cdot \int v'(rk_s^*(w) + A\rho(w)k_r^*(w))\, dP(A) > 0. \qquad \text{(A.36)}$$

By the definition of $R(w)$, (A.36) is equivalent to the inequality

$$-\Delta_t \cdot u'(c_1^*(w)) + \frac{1}{R(w)} \cdot \Delta_{t+1} \cdot u'(c_1^*(w)) > 0$$

or

$$\frac{1}{R(w)} \cdot \Delta_{t+1} > \Delta_t. \qquad \text{(A.37)}$$

Because the new allocation is *interim* Pareto-preferred to the initial allocation, the inequality (A.37) must hold for all relevant wage rates w, in particular for all wage rates in the support of the distribution G_t. The assumption that $G^*(\{w \geq 0 | R(w) > 1\}) > 0$ implies that, for t sufficiently large, $G_t(\{w \geq 0 | R(w) > 1\}) > 0$. Hence, if (A.37) holds for all wage rates in the support of G_t, there exists $\varepsilon > 0$ such that

$$\frac{1}{1+\varepsilon} \cdot \Delta_{t+1} > \Delta_t$$

for all sufficiently large t. But then, the same argument as in the proof of Proposition 4.1 implies that $\Delta_{t^*} = 0$, contrary to the definition of t^*. As in the previous proof, this contradiction implies that the first statement of Proposition 4.3 is true. The proof of the second statement is step by step the same as the proof of the second statement in Proposition 4.1 and is left to the reader. ■

Proof of Proposition 4.4. If $r > 1$, the claim is trivial. Therefore suppose that $r \leq 1$. Let $\underline{w}$ be the minimum of the support of G^*. The, obviously,

$$\underline{w} = \underline{A} \cdot F_\ell(\underline{k}, \bar{L})$$

and

$$k_r^*(\underline{w}) = \underline{k}.$$

Moreover, $\tilde{k}_t = \underline{k}$ and $\tilde{A}_{t+1} = \underline{A}$ imply $\tilde{k}_{t+1} = \underline{k}$. Thus, the net dividend criterion for efficiency implies

$$r \cdot k_s^*(\underline{w}) + \underline{A} \cdot F_k(k_r^*(\underline{w}), \bar{L}) \cdot k_r^*(\underline{w}) \geq (1+\varepsilon)(k_s^*(\underline{w}) + k_r^*(\underline{w})).$$

Since $r \leq 1$, it follows that

$$\underline{A} \cdot F_k(k_r^*(\underline{w}), \bar{L}) \cdot k_r^*(\underline{w}) \geq (1+\varepsilon) \cdot k_r^*(\underline{w}),$$

hence

$$\underline{A} \cdot F_k(k_r^*(\underline{w}), \bar{L}) \geq 1 + \varepsilon.$$

By the definition of $R(\cdot)$, it follows that

$$R(\underline{w}) \geq 1 + \varepsilon$$

and hence that $R(w) > 1$ for w close to $\underline{w}$. ■

Proposition 4.6 follows by combining the arguments in the text with the arguments given in the proof of Proposition 4.1. The details are left to the reader.

A.3 Proofs for Section 5

To study the impact of fiscal policy, I need a more general formulation of individual behaviour. Instead of the problem of maximizing (A.1) under the constraint (A.2), I consider the problem of maximizing

$$u(c_1) + \int v\left(\hat{\Delta}(w) + r \cdot k_s + (A \cdot \rho + \sigma(w)) \cdot k_r\right) dP(A) \tag{A.38}$$

under the constraint

$$c_1 + k_s + k_r = e + w\bar{L} - \Delta, \tag{A.39}$$

where Δ is the lump sum tax imposed in the first period of a person's life and $\hat{\Delta}(w)$ and $\sigma(w) \cdot k_r$ are the lump sum and specific subsidies for the second period of a person's life.

In principle, the solutions to this more general maximization problem can be given a characterization along the lines of Lemma A.1. However, I will not go into this because I do not need it to prove Propositions 5.7 and 5.9. For this purpose, the following lemma will be sufficient.

Lemma A.3 *For any $\Delta > 0$ and $w \geq 0$, let $c_1(\Delta, w), k_s(\Delta, w)$ be a solution to the problem of maximizing*

$$u(c_1) + \int v\left(\Delta + r \cdot k_s + A \cdot \rho(w) \cdot k_r^*(w)\right) dP(A) \tag{A.40}$$

under the constraint (A.39), where $\rho(w)$ and $k_r^*(w)$ are given by Proposition 3.1. Then, for

$$\sigma(w, \Delta) = \frac{\int v'\left(\Delta + r \cdot k_s(\Delta, w) + A \cdot \rho(w) \cdot k_r^*(w)\right) \cdot A \cdot \rho(w)\; dP(A) - u(c_1(\Delta, w))}{\int v'\left(\Delta + r \cdot k_s(\Delta, w) + A \cdot \rho(w) \cdot k_r^*(w)\right)\; dP(A)} \tag{A.41}$$

and

$$\hat{\Delta}(w, \Delta) = \Delta - \sigma(w, \Delta) \cdot k_r^*(w), \tag{A.42}$$

the triple $c_1(\Delta, w), k_s(\Delta, w), k_r^*(w)$ is a solution to the problem of maximizing (A.38) under the constraint (A.39).

Proof. By standard arguments, it suffices to show that, for some $\lambda > 0$, $c_1(\Delta, w), k_s(\Delta, w), k_r^*(w)$ satisfy the first-order conditions

$$u'(c_1(\Delta, w)) = \lambda, \tag{A.43}$$

$$\int v'\left(\Delta + r \cdot k_s(\Delta, w) + A \cdot \rho(w) \cdot k_r^*(w)\right)\, dP(A) \leq \lambda, \tag{A.44}$$

with equality unless $k_s(\Delta, w) = 0$, and

$$\int v'\left(\Delta + r \cdot k_s(\Delta, w) + [A \cdot \rho(w) + \sigma(w, \Delta)] \cdot k_r^*(w)\right) \cdot [A \cdot \rho(w) + \sigma(w, \Delta)]\, dP(A) = \lambda, \tag{A.45}$$

where I have used (A.42) to simplify the argument of $v'(\cdot)$ on the left-hand side of (A.44) and (A.45).

Because $c_1(\Delta, w), k_s(\Delta, w)$ is a solution to the problem of maximizing (A.40) under the constraint (A.39), (A.43) and (A.44) follow from the first-order conditions for $c_1(\Delta, w), k_s(\Delta, w)$ in that maximization. (A.45) follows from the definition of $\sigma(w, \Delta)$ in (A.41). ■

Lemma A.4 *Let $\{\tilde{w}_t\}_{t=1}^{\infty}, (\tilde{\ell}_r^0, \tilde{c}_2^0), \{(\tilde{c}_1^t, \tilde{k}_s^t, \tilde{k}_r^t, \tilde{\ell}_r^t, \tilde{c}_2^t)\}_{t=1}^{\infty}$ be an equilibrium when there is no fiscal policy. For $\Delta > 0$, let $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ be such that, for some $t^* \geq 1$, for $t < t^*$, $\Delta^t = 0$ and $\hat{\Delta}^{t-1}(w) = \sigma^{t-1}(w) = 0$ for all $w \geq 0$, and for $t \geq t^*$, $\Delta^t = \Delta$, $\hat{\Delta}^{t-1}(\cdot) = \hat{\Delta}(\cdot, \Delta)$, as in (A.42), and $\sigma^{t-1}(\cdot) = \sigma(\cdot, \Delta)$, as in (A.41). Then, if $\Delta > 0$ is sufficiently small, the wage process $\{\tilde{w}_t\}_{t=1}^{\infty}$ and the allocation $(\hat{\ell}_r^0(\Delta), \hat{c}_2^0(\Delta)), \{(\hat{c}_1^t(\Delta), \hat{k}_s^t(\Delta), \hat{k}_r^t(\Delta), \hat{\ell}_r^t(\Delta), \hat{c}_2^t(\Delta))\}_{t=1}^{\infty}$ satisfying*

$$\hat{\ell}_r^{t-1}(\Delta) = \tilde{\ell}_r^{t-1} \quad \text{and} \quad \hat{k}_r^t(\Delta) = \tilde{k}_r^t \quad \text{for all } t, \tag{A.46}$$

$$\hat{c}_1^t(\Delta) = c_1(\Delta, \tilde{w}_t) \quad \text{and} \quad \hat{k}_s^t(\Delta) = k_s(\Delta, \tilde{w}_t) \quad \text{for all } t, \tag{A.47}$$

and

$$\hat{c}_2^{t-1}(\Delta) = \Delta + r \cdot \hat{k}_s^{t-1}(\Delta) + A \cdot \rho(w) \cdot \hat{k}_r^{t-1}(\Delta) \quad \text{for all } t \tag{A.48}$$

correspond to an equilibrium for the fiscal policy $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$.

Proof. The lemma follows immediately from Lemma A.3 and the observation that, with unchanged investments in risky capital, market-clearing wage rates are also unchanged in all periods. ■

Proof of Proposition 5.7. Given the specified *laissez-faire* equilibrium, for any small $\Delta > 0$, let $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ be the fiscal policy specified in Lemma A.4 for $t^* = 1$, and let $(\hat{\ell}_r^0(\Delta), \hat{c}_2^0(\Delta)), \{(\hat{c}_1^t(\Delta), \hat{k}_s^t(\Delta), \hat{k}_r^t(\Delta), \hat{\ell}_r^t(\Delta), \hat{c}_2^t(\Delta))\}_{t=1}^{\infty}$ be the associated equilibrium allocation. If the wage rate in period t takes the value $w \geq 0$, then under the latter allocation, a young person in period t gets the overall expected utility

$$u(c_1(\Delta, w)) + \int v(\Delta + r \cdot k_s(\Delta, w) + A \cdot \rho(w) \cdot k_r^*(w))\, dP(A). \tag{A.49}$$

If $\Delta = 0$, (A.49) coincides with the *laissez-faire* equilibrium expected utility

$$u(c_1^*(w)) + \int v(\Delta + r \cdot k_s^*(w) + A \cdot \rho(w) \cdot k_r^*(w))\, dP(A). \tag{A.50}$$

Moreover, (A.49) is differentiable with respect to Δ, with derivative

$$-u(c_1(\Delta, w)) + \int v'(\Delta + r \cdot k_s(\Delta, w) + A \cdot \rho(w) \cdot k_r^*(w))\, dP(A).$$

At $\Delta = 0$, this derivative is equal to

$$-u(c_1^*(w)) + \int v'(r \cdot k_s^*(w) + A \cdot \rho(w) \cdot k_r^*(w))\, dP(A) \tag{A.51}$$

By the definition of $R(w)$, (A.51) can be rewritten as

$$-(1 - \frac{1}{R(w)}) \cdot u(c_1^*(w)).$$

The proposition follows from the assumption that $R(w) < 1$ for all w. ■

Proof of Proposition 5.9. Given the specified *laissez-faire* equilibrium, let t^* be such that, for $t \geq t^*$,

$$\int u'(c_1^*(w)(\frac{R(w) - 1}{R(w)})dG_t(w) < 0. \tag{A.52}$$

For any small $\Delta > 0$, let $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ be the fiscal policy specified in Lemma A.4 for t^*, and let $(\hat{\ell}_r^0(\Delta), \hat{c}_2^0(\Delta)), \{(\hat{c}_1^t(\Delta), \hat{k}_s^t(\Delta), \hat{k}_r^t(\Delta), \hat{\ell}_r^t(\Delta), \hat{c}_2^t(\Delta))\}_{t=1}^{\infty}$ be the associated equilibrium allocation. For $t < t^* - 1$, the *ex ante* expected utility of generation t is the same as in the laissez-faire equilibrium. For generation $t^* - 1$, with $\Delta > 0$, *ex ante* expected utility in the equilibrium associated with the fiscal policy is higher than under *laissez faire*. For generation $t \geq t^*$, ex ante expected utility in the equilibrium associated with the fiscal intervention is given as

$$\int \left[u(c_1(\Delta, w)) + \int v(\Delta + r \cdot k_s(\Delta, w) + A \cdot \rho(w) \cdot k_r^*(w))\, dP(A)\right] dG_t(w). \tag{A.53}$$

If $\Delta = 0$, (A.53) coincides with the *laissez-faire* equilibrium expected utility

$$\int \left[u(c_1^*(w)) + \int v(\Delta + r \cdot k_s^*(w) + A \cdot \rho(w) \cdot k_r^*(w))\, dP(A)\right] dG_t(w).$$

Moreover, (A.53) is differentiable with respect to Δ, with derivative

$$\int \left[-u(c_1(\Delta, w)) + \int v'(\Delta + r \cdot k_s(\Delta, w) + A \cdot \rho(w) \cdot k_r^*(w))\, dP(A)\right] dG_t(w).$$

At $\Delta = 0$, this derivative is equal to

$$\int \left[-u(c_1^*(w)) + \int v'(r \cdot k_s^*(w) + A \cdot \rho(w) \cdot k_r^*(w)) \, dP(A) \right] dG_t(w). \quad \text{(A.54)}$$

By the definition of $R(w)$, (A.54) can be rewritten as

$$\int \left[-u(c_1^*(w)) + \frac{1}{R(w)} u' c_1^*(w)) \right] dG_t(w). \quad \text{(A.55)}$$

By (A.52), it follows that (A.55), and therefore (A.54), is strictly positive. The fiscal policy provides generation $t \geq t^*$ with strict increases in *ex ante* expected utilities. ■

Turning the proof of Proposition 6.10, I first consider a generalization of Lemma A.1.

Lemma A.5 *For any $w \geq 0$, any $\rho > 0$, and any sufficiently small $\Delta > 0$, the problem of choosing c_1, k_s, k_r to maximize*

$$u(c_1) + \int v\left(\Delta + r \cdot k_s + A \cdot \rho \cdot k_r\right) dP(A) \quad \text{(A.56)}$$

subject to the constraint

$$c_1 + k_s + k_r = e + w\bar{L} - \Delta \quad \text{(A.57)}$$

has a unique solution $(c_1(w, \rho, \Delta), k_s(w, \rho, \Delta), k_r(w, \rho, \Delta))$. The solution depends continuously on w, ρ, and Δ. For given Δ, the sections $c_1(\cdot, \cdot, \Delta), k_s(\cdot, \cdot, \Delta), k_r(\cdot, \cdot, \Delta)$ of the functions c_1, k_s, k_r that are determined by Δ have the properties listed in Lemma A.1. In addition, if $r < 1$ and $k_s(w, \rho, \Delta) > 0$, then

$$\frac{\partial k_r}{\partial \Delta}(w, \rho, \Delta) \geq 0, \quad \text{(A.58)}$$

and the inequality is strict if $v(\cdot)$ exhibits strictly decreasing absolute risk aversion. If $k_s(w, \rho, \Delta) = 0$, then, regardless of r,

$$\frac{\partial k_r}{\partial \Delta}(w, \rho, \Delta) < 0. \quad \text{(A.59)}$$

Proof. The first part of the proof is step by step the same as the proof of Lemma A.1 and is left to the reader. One easily verifies that, if $k_s(w, \rho, \Delta) > 0$, then

$$\frac{\partial c_1}{\partial \Delta}(w, \rho, \Delta) = (1 - r) \cdot \frac{1}{\bar{L}} \cdot \frac{\partial c_1}{\partial w}(w, \rho, \Delta)$$

and

$$\frac{\partial k_r}{\partial \Delta}(w, \rho, \Delta) = (1 - r) \cdot \frac{1}{\bar{L}} \cdot \frac{\partial k_r}{\partial w}(w, \rho, \Delta),$$

i.e. the impact on c_1 and k_r of an increase in Δ is the same as the impact of an increase in the wage income $w\bar{L}$ by $(1-r)$ times the increase in Δ. (A.58) thus follows from Lemma A.1 (and the observation that the inequality (A.18) is strict if risk aversion is strictly decreasing). If $k_s(w,\rho,\Delta)=0$, (A.59) follows by applying the implicit function theorem to the first-order condition

$$u'(c_1) = \int v'(\Delta + A\rho k_r)A\rho dP(A)$$

and the budget constraint (A.57). One thereby obtains

$$\frac{\partial k_r}{\partial \Delta}(w,\rho,\Delta) = -\frac{u'' + \int v'' A\rho dP(A)}{u'' + \int v''(A\rho)^2 dP(A)} < 0.$$

■

Lemma A.6 *For any $w \geq 0$ and any sufficiently small $\Delta > 0$, there exists a unique $\bar{k}(w,\Delta) > 0$ such that the function $k_r(\cdot,\cdot,\cdot)$ in Lemma A.5 satisfies*

$$k_r(w, F_k(\bar{k}(w,\Delta),\bar{L}),\Delta) = \bar{k}(w,\Delta). \tag{A.60}$$

The proof of Lemma A.6 is step by step the same as the proof of Lemma A.5 and is left to the reader.

Proof of Proposition 6.10. By the same argument as in the proof of Proposition 3.1, the function $k_r^*(\cdot,\cdot)$ is given by the function $\bar{k}(\cdot,\cdot)$ in Lemma A.6. By (A.60), it follows that, for any $w \geq 0$ and any sufficiently small $\Delta > 0$,

$$\frac{\partial k_r^*}{\partial \Delta} = \frac{\partial k_r}{\partial \rho} \cdot F_{kk} \cdot \frac{\partial k_r^*}{\partial \Delta} + \frac{\partial k_r}{\partial \Delta}, \tag{A.61}$$

where k_r is the function given by Lemma A.5 and the derivative F_{kk} is evaluated at the point $(k_r^*(w,\Delta),\bar{L})$. By Lemma A.5, $\frac{\partial k_r}{\partial \rho} \geq -\frac{k_r}{\rho}$. Therefore, $\frac{\partial k_r^*}{\partial \Delta} < 0$ implies

$$\frac{\partial k_r^*}{\partial \Delta} \geq -\frac{k_r}{\rho} \cdot F_{kk} \cdot \frac{\partial k_r^*}{\partial \Delta} + \frac{\partial k_r}{\partial \Delta}$$

or, since $\rho = F_k(k_r^*(w,\Delta),\bar{L})$,

$$\frac{F_k + k_r F_{kk}}{F_k} \cdot \frac{\partial k_r^*}{\partial \Delta} \geq \frac{\partial k_r}{\partial \Delta}.$$

By the assumption that $F_k + k_r F_{kk} > 0$, it follows that $\frac{\partial k_r^*}{\partial \Delta} < 0$ implies $\frac{\partial k_r}{\partial \Delta} < 0$ and therefore, by Lemma A.5, that $r \geq 1$ or $k_s^*(w,\Delta) = 0$.

By a precisely parallel argument, which I leave to the reader, one also finds that $\frac{\partial k_r^*}{\partial \Delta} > 0$ implies $\frac{\partial k_r}{\partial \Delta} > 0$ and hence, by Lemma A.5, that $k_s^*(w,\Delta) > 0$. Finally, also $\frac{\partial k_r^*}{\partial \Delta} = 0$ implies $\frac{\partial k_r}{\partial \Delta} = 0$. The proposition follows immediately. ■

B Price Effects of Crowding Out

In this appendix, I consider the welfare implications of price effects of changes in risky investments that are caused by a tax-and-transfer scheme without incentive-neutralizing specific subsidies. As indicated by Proposition 6.10, the changes in risky investments involve crowding in or crowding out, depending on whether safe investments under *laissez faire* are positive or not. This finding by itself creates a certain ambiguity about welfare effects, but there is more.

Consider a fiscal policy $\{\Delta^t, \hat{\Delta}^{t-1}(\cdot), \sigma^{t-1}(\cdot)\}_{t=1}^{\infty}$ such that, for some $t^* \geq 1$ and some $\Delta > 0$, $\Delta^t = 0$ for $t < t^*$, $\Delta^t = \Delta$ for $t \geq t^*$, and, moreover, $\hat{\Delta}^t(w) \equiv \Delta$, and $\sigma^{t-1}(w) \equiv 0$ for all t. Given this fiscal policy, using Lemmas A.5 and A.6, one can easily adapt the proof of Proposition 3.1 to show that, if Δ is close to zero, the characterization of equilibrium given there remains valid, with Δ as an additional argument of the functions listed in that proposition. For $t \geq t^*$, the equilibrium levels of first-period consumption, safe and risky investments in period t now take the form

$$\tilde{c}_1^t = c_1^*(\tilde{w}_t, \Delta), \quad \tilde{k}_s^t = k_s^*(\tilde{w}_t, \Delta), \quad \tilde{k}_r^t = k_r^*(\tilde{w}_t, \Delta). \tag{B.1}$$

The wage process can be written as:

$$\tilde{w}_{t+1} = \psi(\tilde{A}_{t+1}, \tilde{w}_t, \Delta) = \varphi(\tilde{A}_{t+1}, k_r^*(\tilde{w}_t, \Delta)) = \tilde{A}_{t+1} \cdot F_\ell(k_r^*(\tilde{w}_t, \Delta), \bar{L}), \tag{B.2}$$

and one obtains

$$\frac{\partial \psi}{\partial \Delta}(\tilde{A}_{t+1}, \tilde{w}_t, 0) = \tilde{A}_{t+1} \cdot F_{\ell k}(k_r^*(\tilde{w}_t, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0). \tag{B.3}$$

Since $F_{\ell k}(k_r^*(\tilde{w}_t, 0), \bar{L}) > 0$, a reduction of $k_r^*(\tilde{w}_t, \Delta)$, lowers the wage rate $\tilde{w}_{t+1}$, and an increase in $k_r^*(\tilde{w}_t, \Delta)$, raises the wage rate $\tilde{w}_{t+1}$, regardless of what the productivity shock $\tilde{A}_{t+1}$ may be. The counterpart of the change in the wage rate is a change in the rate of return

$$\tilde{A}_{t+1} \cdot \rho(\tilde{w}_t, \Delta) = \tilde{A}_{t+1} \cdot F_k(k_r^*(\tilde{w}_t, \Delta), \bar{L}) \tag{B.4}$$

on risky investments, with

$$\tilde{A}_{t+1} \cdot \frac{\partial \rho}{\partial \Delta}(\tilde{w}_t, 0) = \tilde{A}_{t+1} \cdot F_{kk}(k_r^*(\tilde{w}_t, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0). \tag{B.5}$$

Impossibility of Interim Pareto Improvements from Price Effects. A person born in period t is affected by the price effect that the change $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0)$ in the investment $\tilde{k}_r^{t-1}$ in period $t-1$ may have on the person's wage rate $\tilde{w}_t$ in the first period of this person's life and by the price effect that the change $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0)$ in current investment $\tilde{k}_r^t$ may have on the rate of return to the person's risky investments. From an interim perspective, conditioning on the information available to this person, the impact of the price effects of a marginal increase in Δ, starting from $\Delta = 0$, on the person's expected utility is equal to

$$u'(\tilde{c}_1^t) \cdot \tilde{A}_t \cdot \bar{L} \cdot F_{\ell k}(k_r^*(\tilde{w}_{t-1}, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0) \tag{B.6}$$
$$+E[v'(\tilde{c}_2^t) \cdot \tilde{A}_{t+1}|\tilde{w}_t] \cdot k_r^*(\tilde{w}_t, 0) \cdot F_{kk}(k_r^*(\tilde{w}_t, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0).$$

The first term in (B.6) represents the first-period wage effect, the second term the second-period rate-of-return effect. If $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0)$ and $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0)$ have opposite signs, the sign of (B.6) is unambiguous, positive if $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) < 0$ and negative if $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) > 0$. If $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0)$ and $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0)$ have the same sign, the two terms on (B.6) have opposite signs, and the overall assessment of the price effects by a person born in period t depend on which term dominates and on whether there is crowding out or crowding in.

To get a sense of what this assessment is about, it is useful to rewrite (B.6) using the first-order conditions for the choices of $\tilde{c}_1^t$ and $\tilde{k}_r^t$, $u'(\tilde{c}_1^t) = E[v'(\tilde{c}_2^t) \cdot \tilde{A}_{t+1}|\tilde{w}_t] \cdot F_k(k_r^t, \bar{L})$, and the equation $\bar{L} \cdot F_{\ell k}(k_r^*(\tilde{w}_{t-1}, 0), \bar{L}) = -k_r^*(\tilde{w}_{t-1}, 0) \cdot F_{kk}(k_r^*(\tilde{w}_{t-1}, 0), \bar{L})$, which holds because, under the constant-returns-to-scale assumption, the marginal product function $F_k(\cdot, \cdot)$ is homogeneous of degree zero.

$$-u'(\tilde{c}_1^t) \cdot \tilde{A}_t \cdot F_k(k_r^*(\tilde{w}_{t-1}, 0), \bar{L}) \cdot \Phi(\tilde{w}_{t-1}) + u'(\tilde{c}_1^t) \cdot \Phi(\tilde{w}_t), \tag{B.7}$$

where, for any w,

$$\Phi(w) := \frac{k_r^*(w, 0) \cdot F_{kk}(k_r^*(w, 0), \bar{L})}{F_k(k_r^*(w, 0), \bar{L})} \cdot \frac{\partial k_r^*}{\partial \Delta}(w, 0). \tag{B.8}$$

The overall assessment of the price effects of the fiscal intervention by a person born in period t depends not only on the wage rate $\tilde{w}_t$ that this person faces in period t, but also on the wage rate $\tilde{w}_{t-1}$ in the preceding period and the value $\tilde{A}_t$ of the current productivity parameter. The wage rate $\tilde{w}_t = \psi(\tilde{A}_t, \tilde{w}_{t-1}, 0)$ might have come about because $\tilde{w}_{t-1}$ and $k_r^*(\tilde{w}_{t-1}, 0)$ were small and $\tilde{A}_t$ was large or because $\tilde{w}_{t-1}$ and $k_r^*(\tilde{w}_{t-1}, 0)$ were large and $\tilde{A}_t$ was small. In the first case, the absolute value of the first term in (B.7) would be large, in the second case, it would be small. If this impact of history is large enough to reverse the sign, of (B.7), the attitude that a person born in period t has to the price effects of the fiscal intervention actually depends on $\tilde{w}_{t-1}$ and $\tilde{A}_t$. This observation yields the following result.

Proposition B.1 *Assume that $\frac{\partial k_r^*}{\partial \Delta}(w, 0)$ has the same sign for all w. Assume also that, for some $\hat{w}$ in the support of the invariant distribution G^*,*

$$\underline{A} \cdot F_k(k_r^*(w^1(\hat{w}), 0), \bar{L}) < \frac{\Phi(\hat{w})}{\Phi(w^1(\hat{w}))} \tag{B.9}$$

and

$$\bar{A} \cdot F_k(k_r^*(w^2(\hat{w}), 0), \bar{L}) < \frac{\Phi(\hat{w})}{\Phi(w^2(\hat{w}))}, \tag{B.10}$$

where $w^1(\hat{w})$ and $w^2(\hat{w})$ are defined so that

$$\hat{w} = \psi(\underline{A}, w^1(\hat{w}), 0) = \psi(\bar{A}, w^2(\hat{w}), 0).$$

Then, for $\tilde{w}_t = \hat{w}$, the sign of expression (B.6) depends on the value of the $(\tilde{A}_t, \tilde{w}_{t-1})$ and the price effects of the fiscal intervention without incentive-neutralizing subsidies do not provide for an interim Pareto improvement.

In the case of crowding out, i.e., $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0) < 0$ and $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) < 0$, (B.6) is positive if the value of $\tilde{A}_t$ is very small and negative if the value of $\tilde{A}_t$ is very large. Because attitudes towards the allocative changes induced by the price effects depend on the value of $\tilde{A}_t$, these changes cannot provide for an *interim* Pareto improvement in the sense that all participants welcome them, regardless of the information they have, including the information about history. If the maximum $\bar{A}$ of the support of the distribution P is large enough, the harm that a person born in period t suffers from this price effect may even outweigh the benefit that the tax-and-transfer scheme as such would provide in the absence of crowding out and price effects.

The formulation of this result is not quite satisfactory because it makes assumptions about the endogenous functions $w \longmapsto \frac{\partial k_r^*}{\partial \Delta}(w, 0)$, which appears in the premise that there is uniform crowding out or uniform crowding in, as well as the function $w \longmapsto \Phi(w)$, which determines the ratios on the right-hand sides of (B.9) and (B.10).[34] Whereas it is tempting to treat (B.9) and (B.10) as assumptions about the exogenous bounds $\underline{A}$ and $\bar{A}$ of the range of the random variables $\tilde{A}_t$, one must take into account that any assumption about these bounds is an assumption about the probability distribution P, which in turn affects the map $w \longmapsto \frac{\partial k_r^*}{\partial \Delta}(w, 0)$. I use the formulation anyway because it hardly seems worth trying to disentangle the dependence of $\frac{\partial k_r^*}{\partial \Delta}(w, 0)$ on the bounds $\underline{A}$ and $\bar{A}$. The basic insight that no unanimity of assessments, regardless of histories, is to be expected, seems simple enough.

To understand this lack of unanimity, it is useful to take another look at the wage effect $\tilde{A}_t \cdot \bar{L} \cdot F_{\ell k}(k_r^*(\tilde{w}_{t-1}, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0)$ in (B.6). This term shows that, from the perspective of period $t-1$, when the risky investment $k_r^*(\tilde{w}_{t-1}, \Delta)$ is chosen, the choice taken affects not only the distribution of period t incomes between generations $t-1$ and t, capital owners and wage earners, but also the risk exposures of these two parties. In particular, if $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0) < 0$, crowding out reduces the exposure of generation t to risk from the productivity parameter $\tilde{A}_t$. From the perspective of period t, however, the realization of this risk is known. From this perspevtive, the risk reduction from crowding out is undesirable if the value of $\tilde{A}_t$ happens to be very large and desirable if this value

[34] By part (c) of Proposition 3.2, the other term in the definition of $\Phi(w)$, $\frac{k_r^*(w,0) \cdot F_{kk}(k_r^*(w,0), \bar{L})}{F_k(k_r^*(w,0), \bar{L})}$, is bounded away from 0 and -1. The proof of Proposition 3.2 indicates that the bounds depend on the distribution P only through the support minimum $\underline{A}$ and the mean A^*.

happens to be very small. The dependence of the of expression (B.7) on the value of $\tilde{A}_t$ is an instance of the general principle that, once the realization of a risk is known, a person who observes a good outcome considers prior risk sharing a bad thing, and a person who observes a bad outcome considers prior risk sharing a good thing. The interim perspective, which conditions welfare assessments on $\tilde{A}_t$, is not a good basis for assessing the allocation of risks inherent in $\tilde{A}_t$.

Blanchard (2019) approaches the assessment of (B.7) differently. In my notation, he imposes the "steady-state" assumptions $k_r^*(\tilde{w}_t, 0) = k_r^*(\tilde{w}_{t-1}, 0)$ and $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) = \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0)$, implying that $\Phi(\tilde{w}_{t-1}) = \Phi(\tilde{w}_t)$, and he observes that, with crowding out, i.e., if $\Phi(\tilde{w}_{t-1}) = \Phi(\tilde{w}_t) > 0$, the sign of expression (B.7) is positive or negative depending on whether $\tilde{A}_t \cdot F_k(k_r^*(\tilde{w}_{t-1}, 0), \bar{L}) < 1$ or $\tilde{A}_t \cdot F_k(k_r^*(\tilde{w}_{t-1}, 0), \bar{L}) > 1$. He interprets this observation as implying that the price effects of crowding out are (Pareto) beneficial if the rate of return on risky investments is less than the growth rate and detrimental if the rate of return on risky investments is greater than the growth rate.[35]

I have several difficulties with this interpretation. First, the "steady-state" assumption that $k_r^*(\tilde{w}_t, 0) = k_r^*(\tilde{w}_{t-1}, 0)$ and $\frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_t, 0) = \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0)$ presumes that $\tilde{w}_t = \tilde{w}_{t-1}$, so for any $\hat{w}$, there is a unique $\hat{A}$ such that $\tilde{w}_{t-1} = \hat{w}$ implies $\tilde{w}_t = \psi(\hat{A}, \tilde{w}_{t-1}) = \hat{w}$. In a setting with aggregate risk, the "steady-state" assumption singles out particular realizations without giving a perspective on the overall stochastic process or the overall allocation. Second, a welfare assessment of crowding out on the basis of rates of return on risky investments should be formulated in terms of return random variables, rather than specific realizations of these random variables. Third, the term $\tilde{A}_t \cdot F_k(k_r^*(\tilde{w}_{t-1}, 0), \bar{L})$ is *not* a return to an investment of a person born in period t, whose welfare is being considered. This term appears in (B.7) *only* because, under constant returns to scale, the term for the wage effect of crowding-out in (B.6), $\tilde{A}_t \cdot \bar{L} \cdot F_{\ell k}(k_r^*(\tilde{w}_{t-1}, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0)$, happens to be equal in size and opposite in sign to $\tilde{A}_t \cdot k_r^*(\tilde{w}_{t-1}, 0) \cdot F_k(k_r^*(\tilde{w}_{t-1}, 0), \bar{L}) \cdot \frac{\partial k_r^*}{\partial \Delta}(\tilde{w}_{t-1}, 0)$. From the perspective of the person whose welfare is being considered the dependence of this term on $\tilde{A}_t$ reflects the fact that the marginal impact of crowding out on the person's wage rate $\tilde{w}_t$ depends on the combination of $\tilde{w}_{t-1}$ and $\tilde{A}_t$ that generate this wage rate.

***Ex ante* Assessments of Price Effects** Given that price effects involve the extent of risk sharing between succeeding generations, a welfare assessment of price effects should be based on an *ex ante* perspective. The *ex ante* expected

[35] See Blanchard (2019, pp. 1210 f.). His formula (5) is the same as (B.7) above, except that, in going from (B.6) to (B.7), I have eliminated the term involving $v'(\tilde{c}_2^t)$ whereas he eliminated the term $u'(\tilde{c}_1^t)$.

value of (B.7) is given as

$$-\int\int u'(c_1^*(\psi(A_t, w_{t-1}), 0)) \cdot A_t \cdot F_k(k_r^*(w_{t-1}, 0), \bar{L}) \cdot \Phi(w_{t-1})\ dP(A_t) dG_{t-1}(w_{t-1})$$
$$+\int u'(c_1^*(w_t, 0)) \cdot \Phi(w_t)\ dG_t(w_t). \tag{B.11}$$

If t is large, so that both G_{t-1} and G_t are close to G^*, (B.11) is approximately equal to

$$-\int\int u'(c_1^*(\psi(A, w), 0)) \cdot A \cdot F_k(k_r^*(w, 0), \bar{L})\ dP(A)\ dG^*(w)$$
$$+\int u'(c_1^*(w, 0)) \cdot \Phi(w)\ dG^*(w). \tag{B.12}$$

In this *ex ante* formulation, the terms $A \cdot F_k(k_r^*(w, 0), \bar{L})$ are weighted by the marginal utilities $u'(c_1^*(\psi(A, w), 0))$. Because of risk aversion, these terms cannot simply be replaced by their (conditional) means $A^* \cdot F_k(k_r^*(w, 0), \bar{L})$. Indeed, if we had $\Phi(w) > 0$ for all w, the case of crowding out,

$$-\int\int u'(c_1^*(\psi(A, w), 0)) \cdot A^* \cdot F_k(k_r^*(w, 0), \bar{L})\ dP(A)\ dG^*(w)$$
$$+\int u'(c_1^*(w, 0)) \cdot \Phi(w)\ dG^*(w) = 0, \tag{B.13}$$

i.e., if an *ex ante* assessment based on replacing the productivity parameter A by their mean A^* were to indicate indifference, the correct welfare assessment (B.12) would be strictly positive because the reduction of the exposure of generation t to the productivity shock of period t would improve the risk allocation.

This being said, expression (B.12) also indicates that the *ex ante* assessment of price effects from a fiscal policy must also take account of correlations with previous wage rates and previous capital investments. Ultimately, there seems to be no simple criterion by which to assess the *ex ante* welfare impact of these price effects, regardless of whether they involve crowding out or crowding in. In particular, there is no simple criterion related to the return random variables $\tilde{A}_t \cdot F_k(k_r^*(\tilde{w}_{t-1}, 0), \bar{L})$.

The underlying reason for this quandary lies in the lack of *ex ante* Pareto efficiency of the risk allocation under *laissez faire* that stems from the incompleteness of the market system. Given the overlapping-generations structure of the model, the incompleteness of the market system and the inefficiency of risk sharing from an *ex ante* perspective are unavoidable. However, this inefficiency has little to do with dynamic inefficiency. Moreover, the welfare analysis of fiscal interventions with specific subsidies that neutralize crowding out effects shows that our inability to say much about how to improve the risk allocation *ex ante* need not impede our thinking about means to deal with dynamic inefficiency.

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