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Capital Theory of the Steady State

Or: $T = Z - D$

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Abstract: The paper is motivated by the conundrum of the "savings glut" or "secular stagnation". I look at a closed economy with a given nominal rate of growth g in steady state. By an appropriate choice of g we may look at it as the real rate of growth. I compare different steady states each of which is characterized by a steady state rate of interest r. I describe the economy by means of a set of assumptions, which together form the "meta-model". Special cases of the meta-model for the production system of the economy are a Solow macro-economic production function. Or a model with vintage capital. Or a model with endogenous technical progress. Many more models fit the meta-model. If the rate of interest r is a correct price signal for intertemporal choice then I can derive a Generalized Golden Rule of Accumulation. If the government can borrow at the risk-free equilibrium rate of interest r then the Golden Rule optimum $(r = g)$ is characterized by the fundamental equation of steady state capital theory: T$= Z - D$; where Z is the "waiting period" of the representative private household; T is the Böhm-Bawerk period of production, as modernized by Hicks; D is the "public debt period", i.e. net public debt divided by national consumption. An enlarged meta-model includes (land-, oligopoly- etc.-) rents. Let L be their capitalized value divided by national consumption. Then, at the Golden Rule optimum, the extended fundamental equation of steady state capital theory holds: T+$L = Z - D$.

Using this temporal approach in capital theory I introduce two coefficients of intertemporal substitution (CIS): one for the production system and one for the consumption system. For the meta-model, I prove a "law of intertemporal substitution" in terms of CIS. I develop a second degree Taylor approximation for the percentage welfare losses due to a steady state rate of interest different from the rate of growth. They are in proportion to the two CIS in the production system and in the consumption system; in proportion to the square of T in the production system and to the square of Z in the consumption

system, and in proportion to the square of the difference between the rate of interest and the rate of growth. Because the savings glut is due to Z being substantially larger than T the welfare loss is mainly due to $(r - g)$ – induced misallocation in the consumption system and to a much smaller degree due to $(r - g)$ – induced misallocation in the production system. With some calibration of the variables I infer that zero net national debt, $D = 0$, leads to a welfare loss between 25 percent and 45 percent of the maximum real income at the point $r = g$, depending on the degree of risk aversion μ. On top comes the risk of ineffective employment policy due to the zero lower bound.

I investigate potential deviations of a realistic magnitude from the assumptions of the meta-model: 1. The "waiting period" Z rises over time due to rising life expectation. 2. The rate of interest r may be a biased price signal for the production system. 3. There is a negative or positive secular trend in the rate of growth of real income. Due to some of these deviations the optimal steady state rate of interest diverges from the rate of growth. Yet this divergence remains rather small. This result is due to the same fact of life that also leads to the "savings glut": in the 21st century Z is substantially larger than T.

JEL Classification: B22, E69, E12, E13, E14

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Chapter 1 Say´s Law

Under the entry of "Say´s Law" Wikipedia quotes the following statement by J.B. Say: "A product is no sooner created, than it, from that instant, affords a market for other products to the full extent of its own value."[

Neoclassical economics would add: "given that relative prices are in accordance with general equilibrium." This addendum, in a sense, is enough to cover what Say as well as his critics, in particular John Maynard Keynes and his followers, intended to say. To make this contention precise it is useful to investigate a closed economy growing in a steady state. We begin by assuming that there is no technical change. Later in the text, we make clear that we can incorporate many forms of technical change.

The important point is intertemporal relative prices. We can imagine that the cash price of any given consumption good remains constant through time- as production technology does not change. However, there is the real rate of interest, which is well defined, if the commodity basket of the representative consumer remains the same through time. A higher rate of interest means that relative prices of future goods are lower in terms of prices of present goods. Therefore, we may assume that there exists a real rate of interest so that with this rate all product markets and all input markets clear. With this equilibrium rate of interest, we have full employment. Following tradition we give it the name "natural rate of interest". We may assume an overlapping generations´ model. Each generation lives a finite duration of time. Ignoring for the moment bequests to a younger generation we can say that – in terms of present values - each individual consumes as much as he/she "produces, the latter in the form of the "wages" he/she gets for this production.

Assuming for a moment a real rate of interest of zero, Say´s law may be in trouble because each person may consume so late in life or may work so early in life that at any given calendar time the amount of wealth hypothetically owned by private households in general equilibrium exceeds the amount of capital which would be employed by the production system. The answer then is: the natural real rate of interest is negative. Here the Keynesian denial of Say´s law enters the picture: in an economy with cash money there is a zero lower bound of the nominal rate of interest. In this case, without inflation, the "natural rate of interest" cannot be reached. We then encounter the "savings glut problem".

If we do have the savings glut problem, there exist two solutions for this problem. They are: 1. get rid of the zero lower bound and 2. get rid of the savings glut itself. Both solutions can again be provided in two different forms: 1A: get rid of cash as a means of exchange;   1B: find a way to stimulate inflation enough so that the nominal zero lower bound is compatible with the negative real natural rate of interest. If one wants to get rid of the savings glut, there are two ways to do this (or a mixture of the two): 2A: punish savings enough by means of the tax system so that people despair of providing so much for their own future. 2B: let the government incur a sufficient quantity of public debt so that the equilibrium real rate of interest is no longer negative, because the economy at large saves much less than does its private sector.

The following provides a steady state analysis in the spirit of capital theory to shed some light on the savings glut problem and its solutions. Before starting the formal analysis we refer selectively to some policy proposals. Concerning solution 1 (get rid of the zero lower bound) we have Rogoff´s proposal to abolish cash as a means of payment. (Rogoff 2016). We have an earlier proposal by the (then) Chief Economist of the IMF, Blanchard, to try to implement a planned rate of inflation of 4 percent per annum, which would allow a real rate of interest of -4 percent per annum, if needed for full employment (Blanchard et al 2010). Concerning solution 2 (get rid of the savings glut itself), there are people who suggest that a more progressive income taxation will lower the average propensity to save. Alternatively it has been suggested that one should raise public debt within the constraint that the long run real rate of interest for safe assets remain below the long run rate of growth. This solution has been proposed by myself (von Weizsäcker 2010, von Weizsäcker 2o11, von Weizsäcker 2014, von Weizsäcker 2015), and, to my understanding, it is implied by the secular stagnation hypothesis of Summers (Summers 2013, Summers 2014, Rachel and Summers 2019) and by Blanchard´s analysis in his AEA-Presidential Address (Blanchard 2019).

Chapter 2 The Standard “Meta-Model”.

Section 1. The Generalized Golden Rule of Accumulation.

This is the philosophy of my modelling approach: I want to concentrate on the steady state of a closed economy in the tradition of capital theory, as it originated in Böhm-Bawerk 1889 and continued to be further developed for almost a century up until the year 1980. By exploiting the assumption that we look at a steady state one can derive results without having to specify all the details of the underlying model in terms of production and consumption. For the beginning the reader may assume that 1) the available technology of the economy remains constant through time, 2) households have fixed preferences, which are the same from one household to the next one, 3) we work in an overlapping generations´ model. As we shall see later, we can generalize these assumptions and still get the same results. Thus, technical change, even certain forms of endogenous technical change, rising life expectancy and other changes all are compatible with the results of the standard model.

Given that we don´t have to specify all details of the underlying model in terms of production and consumption, I call my model a "meta-model".

We concentrate on the comparison of full employment steady state situations. Each such steady state has a different equilibrium rate of interest, r. The rate of interest remains constant through time. The system at large has an exogenously given rate of growth, g. The best way to understand our approach is to consider variables like income, wages, the interest rate, the rate of growth of the system in nominal terms i.e. in, say, Euros per unit, whatever the unit may be. But, as long as we can unequivocally define a rate of inflation, we may also get hold of real variables like the real rate of growth, the real rate of interest etc. Using certain more specific models in later chapters, we will see the advantages of using nominal variables: in particular, changes in the structure of the available technology and ensuing changes in relative prices of consumption goods can be dealt with in this steady state comparison.

The set of possible rates of return (rates of interest) r for safe assets is a set of real numbers and thus a subset of the set of real numbers. Here we denote the set of possible rates of return, r, by the capital letter R. We then make the following assumption

Assumption 1: The set R is a connected subset of the set of real numbers. The exogenously given growth rate g is contained in R.

We exclude interest rates, which are so high that the system cannot pay any wages. It may thus be the case that R is strictly smaller than the set of real numbers. Moreover the set R may be bounded below. But the connectedness of R is not affected: For any given $r \in R$ any smaller r above its lower bound is also a member of R.

Provided that for any given steady state the rate of interest, r, is given, a particular technique of production θ is in use. This means that in equilibrium each production sector of this multi-commodity economy has chosen its particular technique of production. We do not have to specify the details of the technique in use. We simply say that θ is a function of the prevailing steady state rate of interest, i.e. $\theta = \theta(r)$. Given the steady state rate of interest r, the expression $\theta(r)$ simply is the "name" of the prevailing technique of production. It is then convenient to apply the following "naming convention": $\underline{\theta(r) \equiv r}$. But, of course, the technique $\theta(r)$ must be distinguished conceptually from its "name giver" r.

Let then $Theta$ be the set of techniques θ for which there exists a rate of interest $r \in R$ such that $\theta = \theta(r)$. In a formula

$$Theta = \{\theta: \exists r \in R, s.t. \theta = \theta(r)\}$$

Note that this set $Theta$ is a "small" subset of all feasible techniques of production, because there are likely to be many feasible techniques of production, which will not be applied at any steady state rate of interest.

Now we define y as (nominal) annual net social product per "head" at time zero. By "head" we mean a fully employed average worker in the economy. We also may say: y is the economy´s labor productivity (in nominal terms) at time zero. We now split y into two components

$$y = w(r;\theta) + rv(r;\theta)$$

Here $v(r;\theta)$ is the average amount of capital per worker used in the production process. Thus, $rv(r;\theta)$ is the average cost of capital per worker in this economy. The remaining component $w(r;\theta) = y - rv(r;\theta)$ is the "rest of income" after capital got its return. In capital theory from Böhm-Bawerk´s time on and including the Cambridge-Cambridge controversy this "rest of income" was identified with the wage rate. Thus, looking at $w(r;\theta)$ as a function of r is what Samuelson called the factor price frontier (Samuelson 1962), and what we also may call the "wage-interest-curve". Keeping θ constant is then the wage-interest curve for any given technique. And varying $\theta(r)$ with r then leads to a kind of "envelope" of the different technique-specific wage-interest curves.

However, in our general approach in this paper we do not have to identify $w\ (r;\theta)$ with the average wage rate. It is then, formally, just what remains of value added per worker after capital has its share in the form of the risk-free rate of interest. The models traditionally used in capital theory are just a special case of our "meta-model". I emphasize that $w\ (r;\theta)$ also contains risk premiums, which are paid to capital owners.

Now I introduce the conventional national accounting division of value added into consumption c and net investment I. We can write

$$y = c(r;\theta) + I(r;\theta)$$

Let us now look at the special case that in any steady state the capital/consumption ratio remains constant through time. This amounts to

<u>Assumption 2: For any given steady state rate of interest r net investment $I(r;\theta) = gv(r;\theta)$</u>

Using these two divisions of y we can eliminate y. We have

$$w(r;\theta) + rv(r;\theta) = c(r;\theta) + gv(r;\theta) \qquad (1)$$

We now discuss the nature of the function $c(r;\theta)$. We may first assume that there exists a particular unit commodity basket of consumption goods such that for all $r \in R$ the actual commodity basket consumed is $c(r;\theta)$ times that unit consumption commodity basket. That is, there is no substitution between consumption goods whenever relative prices change due to a change in the rate of interest. In that simple case it is clear that $c(r;\theta)$ only varies with θ. There is no separate influence of r on c. For, θ, the production system, already determines the output vector of consumption goods. So, we then can write $c = c(\theta)$. In that case the unit of account of our system is the unit consumption commodity basket. In this special case, accounting in real terms and accounting in nominal terms coincide. Later, as we turn to the demand side for consumption goods in Section 2 below, we see that our general approach does not need this assumption of non-substitution between consumption goods.

We now turn to our "smoothness" assumption. Using our naming convention $\theta(r) \equiv r$ we assume that variables $w(r;\theta(r))$ and $v(r;\theta(r))$ are "smooth" functions of r and of θ. This is then our

<u>Assumption 3: For $r \in R$ and for $\theta \epsilon Theta$ the values $w(r;\theta)$ and $v(r;\theta)$ are continuously differentiable functions of r and θ. Addendum to Assumption 3:</u>

the variables $\bar{w}(r;\eta(r;\bar{U}))$ and $\bar{v}(r;\eta(r;\bar{U}))$ to be defined in Section 2 also are continuously differentiable functions of their arguments.

This assumption is reasonably realistic. That values like w or v are continuously differentiable with respect r, keeping θ fixed, is quite obviously the case in typical economic models. Continuous differentiability with respect to θ basically means that there is a sufficient "richness" of available techniques.

I now turn to the rate of interest as a price signal.

Assumption 4: (Unbiased price signal r, part 1 (UBPS1)): For each $r \in R$ the following inequality holds

$$w\big(r;\theta(r)\big) \geq w(r;\theta)\ for\ all\ \theta \in Theta$$

This assumption corresponds to routine assumptions in capital theory. But it is of interest even beyond traditional capital theory. The economic meaning is clear: in the competition among different techniques that one prevails which allows achieving a maximal remuneration of the other (non-capital) factors of production. Outside of traditional capital theory, this assumption is substantially weaker than the assumption of efficiency or, which is the same, Pareto optimality. We allow for the possibility that all techniques in the set $Theta$ are hugely inefficient; only such inefficiency then must not affect different steady states to a different degree. Later in the paper we shall replace this assumption by a weaker assumption which is compatible with a bias in the price signal r.

Before we come to our first theorem I introduce one additional assumption. It is the assumption that any small change in the steady state rate of interest goes with some substitution. It takes the following form

Assumption 5: "Law of demand". At any steady state rate of interest, $r\epsilon R$ a marginal rise in the rate of interest induces a change in the production system such that the capital-output ratio declines.

This is simply an application of the "law of demand", here applied to the inputs of the production system. A higher rate of interest means that capital has become more expensive, whereas the remuneration of the other factors of production is lower; the latter can be seen in the fact that w is lower if the rate of interest is higher. Neoclassical production theory predicts that such a change in the relative prices of inputs will be answered by a transition towards methods of production which uses less capital per unit of output.

In a mathematical formula we obtain the following inequality. Output per head is $gv(r;\theta)+c(\theta)$. The inverse of the capital-output ratio then is $\frac{gv(r;\theta)+c(\theta)}{v((r;\theta)}=g+\frac{c(\theta)}{v(r;\theta)}$. We then assume that this expression rises with rising θ, keeping the weighting system of prices, induced by r, the same. So in a formula we have

<u>Assumption 5:</u>

$$\frac{1}{c(\theta)}\frac{dc}{d\theta}>\frac{1}{v(r;\theta)}\frac{\partial v(r;\theta)}{\partial\theta}$$

Using these five assumptions and using our equation

$$w(r;\theta)+rv(r;\theta)=c(r;\theta)+gv(r;\theta)$$

we get a certain inequality which is very useful for our further work. First we observe that because of Assumption 4 (the rate of interest is a correct price signal) we obtain for $\theta=r$ that

$$\frac{\partial w(r;\theta)}{\partial\theta}=0$$

Therefore, we can write

$$\frac{dc}{d\theta}=(r-g)\frac{\partial v}{\partial\theta}$$

which can be written in the following form

$$\frac{1}{c(\theta)}\frac{dc}{d\theta}=\frac{v(r;\theta)}{c(\theta)}(r-g)\frac{1}{v(r;\theta)}\frac{\partial v(r;\theta)}{\partial\theta}$$

Now we know that, due to $w(r;\theta)>0$, the expression $\frac{v(r;\theta)}{c(\theta)}(r-g)<1$.

Together with the inequality of Assumption 5 (the "law of demand") this leads to the following expressions for $\frac{dc}{d\theta}$

Case 1: $r<g$. In that case the logarithmic derivatives for c and for v have opposite signs. Our law of demand then implies

$$\frac{dc}{d\theta}>0 \text{ and } \frac{\partial v(r;\theta)}{\partial\theta}<0$$

As the rate of interest rises marginally consumption per head rises, whereas capital used per head declines (keeping the weighting system, induced by r, the same).

Case 2: $r > g$. In that case the logarithmic derivatives for c and for v have the same sign. Given that $\frac{v(r;\theta)}{c(\theta)}(r-g) < 1$, the inequality of the law of demand can only be fulfilled if both logarithmic derivatives are negative. We then have

$$\frac{dc}{d\theta} < 0 \text{ and } \frac{\partial v(r;\theta)}{\partial \theta} < 0$$

As the rate of interest rises marginally, both, consumption per head and capital used per head decline (keeping the weighting system, induced by r, the same).

Case 3: $r = g$. Our equation and the law of demand then imply

$$\frac{dc}{d\theta} = 0 \text{ and } \frac{\partial v(r;\theta)}{\partial \theta} < 0$$

We then have shown

<u>Theorem 1: "Generalized Golden Rule of Accumulation": Under Assumptions 1 to 5 the following holds: Within the set R steady state consumption obtains its maximum at a rate of interest $r = g$. Moreover, for $r < g$ a marginal rise of the rate of interest raises steady state consumption; for $r > g$ a marginal rise of the rate of interest reduces steady state consumption.</u>

We may express Theorem 1 in the following way: within the set R steady state consumption is a continuously differentiable function of r which is single peaked with its maximum at the rate of interest, which equals the growth rate.

There is a converse to Theorem 1:

<u>Theorem 1 Almost Converse: Assume Assumptions 1,2,3, and 4. If, within the set R steady state consumption obtains its maximum at a rate of interest $r = g$; If moreover, for $r < g$ a marginal rise of the rate of interest raises steady state consumption; if moreover for $r > g$ a marginal rise of the rate of interest reduces steady state consumption then</u>

$$\frac{1}{c(\theta)}\frac{dc}{d\theta} \geq \frac{1}{v(r;\theta)}\frac{\partial v(r;\theta)}{\partial \theta}$$

<u>Proof:</u> Using the equation

$$\frac{1}{c(\theta)}\frac{dc}{d\theta} = \frac{v(r;\theta)}{c(\theta)}(r-g)\frac{1}{v(r;\theta)}\frac{\partial v(r;\theta)}{\partial\theta}$$

and keeping in mind that, due to $w(r;\theta) > 0$, we have $\frac{v(r;\theta)}{c(\theta)}(r-g) < 1$ we can read the equation from left to right to get the result for $r < g$ and $r > g$, in the form of strict inequality. Due to Assumption 3 the weak inequality then also must hold for $r = g$. QED.

The almost converse of Theorem 1 is useful, if in any given specific model satisfying the conditions of the meta-model, we know that $\frac{dc}{d\theta}(r-g) \leq 0$.

As a side remark: Assumptions 1, 2, 3, and 4 also imply that the "marginal productivity of capital" equals the rate of interest r. We keep in mind the equation

$$\frac{\partial w(r;\theta)}{\partial\theta} = 0 \text{ at } \theta = r$$

Let φ be the marginal productivity of capital. We can define it by means of the following equation

$$\varphi = \frac{dc/d\theta + g\,\partial v/\partial\theta}{\partial v/\partial\theta} = \frac{\partial w/\partial\theta + r\,\partial v/\partial\theta}{\partial v/\partial\theta} = r$$

This expression is well-defined for $\partial v/\partial\theta \neq 0$. We have shown above, that Assumptions 1,2,3,4, and 5 imply $\partial v/\partial\theta < 0$.

Section 2. Steady State Consumption for a Given Level of Life Utility

The Generalized Golden Rule of Accumulation refers to the production system of the economy. There is a mirror theorem for the consumption system of the steady state economy. For a special case, this mirror theorem had been announced even before the Golden Rule of Accumulation was published (Phelps 1961; Weizsäcker 1961 and1962): in Samuelson´s famous overlapping generations paper (Samuelson 1958).

We think in terms of an overlapping generations´ model. At any given moment of time there are many age cohorts. As we work in a model with continuous time we so to speak have an "infinity" of age cohorts. To put the same idea differently: at any given moment of calendar time there is a density distribution of age cohorts.

In microeconomics, we know the law of demand, which we already have used above for the production system (Assumption 5 above). However, the law of demand also applies to consumption behavior: keeping real income the same a rise in the price of a consumption good reduces demand for that good. Similarly for the household supply side: keeping real income the same, a rise in the wage rate raises the supply of labor. We now apply this law of demand within a steady state overlapping generations' model.

In the following exercise we assume that the life plan of a person in terms of consumption and labor supply, η, generates some "life utility" U. We may write down an indirect utility function $U(\widehat{w}; r)$. Here r is the prevailing rate of interest and $\widehat{w}$ is the "wage rate" available to the person. For simplicity we may assume that that wage rate $\widehat{w}$ is constant through time. But the theory provided here is quite general, as we shall see in later sections.

Assumption 6 is again, like Assumption 4 an assumption that the rate of interest r is an unbiased price signal; in this case applied to the consumption sector of the economy. Let life utility U be a function of the life plan η of the person: $U = U(\eta)$. Let $\overline{w}(r; \eta)$ be the "wage rate" required to finance pattern η, given the rate of interest *r.* Let $Eta(\overline{U})$ be the set of patterns η such that their life utility $U(\eta)$ is higher than $\overline{U}(\bar{\eta})$. We then make Assumption 6

Assumption 6: (Assumption of unbiased price signal r, part 2 (UBPS2))

$$\overline{w}(r; \bar{\eta}) < \overline{w}(r; \eta) \; for \; all \; \eta \in Eta(\overline{U}(\bar{\eta}))$$

The economic meaning of this assumption is this: the consumer chooses a pattern $\bar{\eta}(r)$ which maximizes life utility under the constraint that he/she can finance it – given the rate of interest *r* and the "wage rate" $\overline{w}(r; \bar{\eta})$ available to him/her: any pattern with a higher life utility is out of reach, given the budget constraint.

In parallel to Assumption 2 we also make Assumption 7. Let $\hat{v}(r; \eta(r))$ be average wealth per head in the overlapping generations´ model, as a function of the rate of interest and the corresponding equilibrium "wage rate" $\widehat{w}(r; \eta(r))$

Assumption 7: The level of savings compatible with maintaining the prevailing steady state rate of interest equals $g\hat{v}(r; \eta(r))$

We then obtain an equation, which is the mirror image of equation (1)

$$\widehat{w}(r; \eta) + r\hat{v}(r; \eta) = \hat{c}(r; \eta) + g\hat{v}(r; \eta) \qquad (2)$$

Here $\widehat{w}(r;\eta)$ is the "wage rate" paid per person; and $\hat{c}(r;\eta)$ is consumption per person. For reasons, which become clear in section 3 below, $\widehat{w}(r;\eta(r))$ need not be equal to $w(r;\theta(r))$.

For the following steady state exercise, we assume life utility $\overline{U}$ of the representative consumer to be given. It is thus an exercise in the spirit of "the law of demand", as it was discussed above. We look at a steady state economy in which the representative consumer obtains a given life utility level $\overline{U}$. Let the corresponding wage rate be denoted by $\overline{w}(r;\eta)$. Given the utility level $\overline{U}$ and given Assumption 6 the labor-consumption pattern η is determined, once we know the rate of interest and the corresponding wage rate $\overline{w}(r;\eta)$. Without loss of generality we can give that pattern η the "name" r; keeping always in mind that the life utility level $\overline{U}$ has been fixed. Thus, we write $\eta(r) = r$. As before in the case of the production system θ we are then able to partially differentiate with respect to η.

We look at a steady state with the given life utility $\overline{U}$ of the representative consumer. Let $\bar{c}(r;\eta)$ be consumption per worker. Let $\bar{v}(r;\eta)$ be wealth per worker. We then obtain the equation

$$\bar{c}(r;\eta) + g\bar{v}(r;\eta) = \overline{w}(r;\eta) + r\bar{v}(r;\eta)$$

It is equation (2) applicable to the case of a fixed level of life utility $\overline{U}$. In the following we investigate partial derivatives of the variables with respect to η. Here we always mean the partial derivative at $\eta = r$. Now we prove the following

<u>Lemma: With Assumption 6 we have for the partial derivative</u> $\frac{\partial \overline{w}}{\partial \eta} = 0$

<u>Proof:</u> Assume the contrary. We then have $\frac{\partial \overline{w}}{\partial \eta} \neq 0$. Thus we can find some $\eta \neq r$ such that $\overline{w}(r;\eta) < \overline{w}(r;r)$. Due to continuity of the function $\overline{w}(r;\eta)$ we then also can find $\bar{\eta}$, such that $\overline{w}(r;\bar{\eta}) < \overline{w}(r;r)$ with $\bar{\eta} \in Eta(\overline{U})$, contrary to Assumption 6. QED.

Using this Lemma we get the following result by partially differentiating our equation with respect to η.

$$\frac{\partial \bar{c}}{\partial \eta} = (r-g)\frac{\partial \bar{v}}{\partial \eta}$$

Now we introduce our version of the "law of demand" and "the law of supply". It is the mirror image in the consumption system of Assumption 5 in the production system.

Assumption 8: $\frac{\partial \bar{c}}{\partial \eta}\frac{1}{\bar{c}} < \frac{\partial \bar{v}}{\partial \eta}\frac{1}{\bar{v}}$

The economic meaning of this assumption is the following: a rise in the return on wealth (=a rise in the rate of interest) compensated by a decline in the wage rate so that life utility remains the same induces a rise in average wealth over the life-time of the representative consumer relative to the average level of consumption. The higher return on wealth induces some postponement of consumption towards later in life, because the relative price of later consumption has fallen whereas the relative price of early consumption has gone up. And the higher return on wealth induces some shifting forward of labor supply, because the relative price of early labor supply has gone up, whereas the relative price of later labor supply has fallen. In this inequality expressing the law of demand (and the "law of supply") the weighting system induced by r has remained constant.

Assumptions 6,7 and 8 lead to the following expressions for $\frac{\partial \bar{c}}{\partial \eta}$

Case 1: $r < g$. Then, by $\frac{\partial \bar{c}}{\partial \eta} = (r-g)\frac{\partial \bar{v}}{\partial \eta}$, the derivatives $\frac{\partial \bar{c}}{\partial \eta}$ and $\frac{\partial \bar{v}}{\partial \eta}$ have opposite signs, which together with Assumption 8 implies

$$\frac{\partial \bar{c}}{\partial \eta} < 0$$

Case 2: $r > g$. We then can write

$$\frac{\partial \bar{c}}{\partial \eta}\frac{1}{\bar{c}} = (r-g)\frac{\bar{v}}{\bar{c}}\frac{\partial \bar{v}}{\partial \eta}\frac{1}{\bar{v}} = \left(1 - \frac{\bar{w}}{\bar{c}}\right)\frac{\partial \bar{v}}{\partial \eta}\frac{1}{\bar{v}}$$

from which follows that $\frac{\partial \bar{c}}{\partial \eta}$ and $\frac{\partial \bar{v}}{\partial \eta}$ have the same sign. But then by the inequality of Assumption 8 and by the inequality $\left(1 - \frac{\bar{w}}{\bar{c}}\right) < 1$, both, $\frac{\partial \bar{c}}{\partial \eta}$ and $\frac{\partial \bar{v}}{\partial \eta}$ have to be positive. So the result is

$$\frac{\partial \bar{c}}{\partial \eta} > 0$$

Case 3: $r = g$. We then, obviously, have

$$\frac{\partial \bar{c}}{\partial \eta} = 0$$

We now introduce the fact that we are at liberty how to define the general price level. We want to define the general price level in such a way that it is consistent with our analysis of steady state consumption in Section 1 above. There we assumed that a changeover from one steady state to another one with a different rate of interest does not change the unit consumption basket, which was our unit of account. This implied that the partial derivative $\frac{\partial c(r;\theta)}{\partial r}$ is zero. To be consistent we then also can assume that, keeping the life utility level constant the steady state consumption $\bar{c}(r;\eta)$ also has a partial derivative with respect to r which is zero. We then assume a definition of the general price level so that

$$\frac{\partial \bar{c}(r;\eta)}{\partial r} = 0$$

But we should note that this assumption is much weaker than the assumption of a non-changing unit basket of consumption goods. Basically we are free how to set the price level of the steady state economy. Thus, without loss of generality, we can set the price levels of the different steady states in such a way that the partial derivative $\frac{\partial \bar{c}(r;\eta)}{\partial r}$ is set to zero.

Taking this approach, we then obtain the result that steady state consumption per head for a given life utility level is at its minimum when the rate of interest equals the rate of growth. But this, together with the Generalized Golden Rule of Accumulation (Theorem 1 above) implies the following

<u>Theorem 2 (Golden Rule of Life Utility): Under Assumptions 1-8, comparing different steady states, life utility of the representative consumer is maximized at the steady state which exhibits a rate of interest equal to the rate of growth.</u>

<u>Proof:</u> We take the life utility corresponding to a consumption level $\bar{c} = c(g;\theta(g))$ of the production system for a rate of interest equal to the rate of growth. Obviously for $r \neq g$ we have $c(r;\theta(r) < c\big(g;\theta(g)\big) = \bar{c}\big(g;\eta(g)\big) < \bar{c}\big(r;\eta(r)\big)$. Therefore $c(r;\theta(r)$ corresponds to a lower life utility than the one obtainable at $r = g$. QED.

Section 3 The public debt period D and the period of production T

So far, we have compared different steady states with different steady state rates of interest. But we have not yet looked at the policy instrument by which the "benevolent dictator" can influence the choice of the steady state rate of interest. That instrument is the level of public debt. We expect that the steady state rate of interest rises with a rising level of public debt. The "natural rate of interest" then is that steady state rate of interest, which goes together with a zero level of net public debt. We give it the "name" Rho or in a Greek letter: ρ.

My approach can be seen as a generalization of Diamond 1965.

Let public debt per head be $Dw(r;\theta)$. We may call D the "public debt period", as it has the dimension "time". We assume that the steady state primary government surplus $(r-g)Dw(r;\theta)$ per head arises by means of a tax on households. The "wage income" of the representative citizen then is given by

$$\widehat{w} = w - (r-g)Dw = w(1-(r-g)D)$$

Obviously, $(r-g)Dw$ can be positive, zero or negative. The public debt period D is a concept similar to the conventionally used public debt ratio: the latter divides public debt by the annual gross national product (or gross domestic product), whereas D divides public debt by the annual steady state "wage income" in nominal terms.

From now on we use the following notation for the partial derivatives. The partial derivatives with respect to r are defined within the set R; the partial derivatives with respect to θ are defined within the set $Theta$ discussed above. We look at the function $w(r;\theta)$. We then write

$$w_1(r;\theta) = \frac{\partial w(r;\theta)}{\partial r};\ \ w_2(r;\theta) = \frac{\partial w(r;\theta)}{\partial \theta};\ w_{11}(r;\theta) = \frac{\partial^2 w(r:\theta)}{\partial r^2};$$
$$w_{12}(r;\theta) = \frac{\partial^2 w(r:\theta)}{\partial r \partial \theta};\ w_{21}(r;\theta) = \frac{\partial^2 w(r:\theta)}{\partial \theta \partial r};\ w_{22}(r;\theta) = \frac{\partial^2 w(r:\theta)}{\partial \theta^2}$$

Due to Assumption 4 (UBPS1: unbiased price signal *r*, part 1) we know that, at any given value r, the partial derivative of *w* with respect to θ at $\theta = r$ is equal to zero. We then can write (envelope theorem)

$$\frac{dw(r;\theta(r))}{dr} = w_1\big(r;\theta(r)\big) + w_2\big(r;\theta(r)\big) = w_1(r;\theta(r)).$$

For the first partial derivative we write

$$w_1(r;\theta) = -w(r;\theta)T(r;\theta) \qquad (3)$$

This new expression *T* has the dimension "time". This is so because the expression $\frac{w_1}{w}$ has the dimension "time". Normally, in economic models w_1 is negative. Hence *T*, as defined, is a positive length of time.

In models, which have been used in capital theory we can derive an economic meaning for *T*. Indeed, as I have shown in Weizsäcker 2011, in such models *T* is the modernized (by Hicks in Value and Capital 1939) version of Böhm-Bawerk´s average period of production (Böhm-Bawerk 1889). See also below Section 5, and again Chapter 3. Böhm-Bawerk claimed that the capital requirement of production per worker is equal to the wage rate times the period of production.

From section 1 (Equation 1) above we can derive an equation for the steady state capital requirement. Indeed, for $r \neq g$ we can write

$$v\big(r;\theta(r)\big) = \frac{c(\theta)-w(r;\theta(r))}{r-g} \qquad \text{(1a)}$$

In order to get the capital requirement for $r = g$ we can apply L´Hopital´s rule.

By L´Hopital´s rule, using the partial derivative with respect to r, we obtain

$$v(g;g) = \frac{-w_1}{1} = T(g;g)w(g;g) \qquad \text{(1b)}$$

I call equation (1b) the <u>"Böhm-Bawerk equation"</u>. Here, at the optimal level of the rate of interest, we then obtain the result, which corresponds to the Böhm-Bawerk theory of capital. In models other than the traditional models of capital theory one would have to show whether *T* as defined here has the same economic meaning. Outside of the optimum rate of interest the capital requirement, as a rule, does not correspond to Tw. In this paper I use the term "period of production" for the mathematically defined expression $T = -w_1/w$.

<u>Section 4. The "waiting period" Z</u>

Ignoring for the moment capitalized rents (like Ricardian land rents, monopoly rents etc.) an equation for personal wealth per head applies:

$$\hat{v}(r) = v\big(r;\theta(r)\big) + Dw(r;\theta) \qquad \text{(2a)}$$

Personal wealth equals capital tied to the production sector per worker plus public debt per worker. In particular, for $r = g$,

$$\hat{v}(g) = v\big(g;\theta(g)\big) + Dw(g;\theta(g) = w(g;\theta(g))(T + D) \qquad \text{(2b)}$$

The public debt period D includes "implicit public debt". In many countries the most important component of "implicit public debt" are future pension claims of people insured in the social security system. In most countries, these claims against the government are not funded on the side of the debtor, i.e. the government. These unfunded debts are then part of net government debt.

Now we take account of the slope of the indifference curves in the $r, \widehat{w}$ –space. The representative household is a net owner of wealth. Thus, a higher rate of interest *r* implies that less "other income" $\widehat{w}$ is needed to maintain the same level of life-time utility. Indeed, as can be shown in a lifetime saving model, we obtain the following first derivative for utility with respect to *r*. Let life utility be defined in terms of an indirect ordinal utility function $U = U(r; \widehat{w})$. For the total derivative with respect to *r* we then obtain the following $\frac{dU}{dr} = U_r + U_{\widehat{w}} \frac{d\widehat{w}}{dr}$. In a lifetime consumption model we can derive a relation between the two partial derivatives:

$$U_r = Z\widehat{w}U_{\widehat{w}}.$$

Here the variable Z has the dimension "time". In a lifetime consumption model we find that Z is what I call the <u>"waiting period"</u>. In present value terms, it is the average time distance between consumption outlays and wage income. Thus, for example, if consumption is spread equally over the life between 20 years of age and 80 years of age, if wage income is equally spread between 20 years of age and 60 years of age, and if the prevailing rate of interest is zero then

$$Z = \tfrac{1}{2}\{(80-20)\ years - (60-20) years\} = 10\ years.$$

Now we can write for any given steady state

$$\frac{dU}{dr} = U_r + U_{\widehat{w}} \frac{d\widehat{w}}{dr} =$$

$$= U_{\widehat{w}} \left\{ Zw(1+(g-r)D) - Tw(1+(g-r)D) - wD + w(g-r)\frac{dD}{dr} \right\} =$$

$$U_{\widehat{w}} w \left\{ Z - T - D + (g-r)D(Z-T) + (g-r)\frac{dD}{dr} \right\}$$

For the special case of the natural rate of interest $r = \rho$, by definition we have $D = 0$; and therefore

$$\frac{dU}{dr} = U_{\widehat{w}} w \left\{ Z - T + (g-\rho)\frac{dD}{dr} \right\}$$

For the special case $r = g$ we get

$$\frac{dU}{dr} = U_{\widehat{w}} w\{Z - T - D\}$$

Now we know from Theorem 2 that steady state life utility obtains its maximum at $r = g$. So there $\frac{dU}{dr} = 0$. But $U_{\widehat{w}} w > 0$. So we obtain

Theorem 3: At $r = g$ we have $Z - T - D = 0$

The economic interpretation of this Theorem is this: We may call $Z - D$ the "waiting period of the economy at large". It is the sum of the waiting period of the private sector, Z, and the (perhaps negative) "waiting period" of the public sector, $-D$. In the steady state optimum the waiting period of the economy is equal to the period of production.

Due to the fact that at the steady state optimum the waiting period at large equals the period of production we give the equation

$$T = Z - D$$

the name: fundamental equation of steady state capital theory

Section 5. A characteristic of the wage-interest curves derived from the division of labor

For our further work in this paper it is useful to have a closer look at the period of production T and at the waiting period Z. For this purpose we look at a production system with a given flow of labor inputs $\lambda(t)$ and a given flow of a consumption good output $\gamma(t)$. Think of a firm, which is hundred percent vertically integrated, so that it only buys labor inputs from outside and only sells the consumption good to the outside. Consider the case that this production system just breaks even. We then have the equation

$$w \int_{-\infty}^{\infty} e^{-rt} \lambda(t) dt - \int_{-\infty}^{\infty} e^{-rt} \gamma(t) dt = 0$$

I assume that we can construct a fully vertically integrated firm so that these integrals have a finite value. Now I differentiate this system with respect to the rate of interest, maintaining the zero profit equation above by an appropriate change of the wage rate.

Let $T_\lambda = \frac{\int_{-\infty}^{\infty} te^{-rt}\lambda(t)dt}{\int_{-\infty}^{\infty} e^{-rt}\lambda(t)dt}$ be the "time point of gravity" of labor inputs

Let $T_\gamma = \frac{\int_{-\infty}^{\infty} te^{-rt}\gamma(t)dt}{\int_{-\infty}^{\infty} e^{-rt}\gamma(t)dt}$ be the "time point of gravity" of consumption good outputs.

Let $T = T_\gamma - T_\lambda$ be the "period of production". It is then easy to show that

$$\frac{dw}{dr}\frac{1}{w} = -T$$

I have used this result already in Section 3 to justify the interpretation of $-w_1/w$ as being the period of production. Note that we have not changed the flow of inputs and the flow of outputs. Thus, substitution effects are not yet in the picture.

Now I differentiate T with respect to r.

Without too much computation effort, the result is this

$$\frac{dT}{dr} = Variance\,\{e^{-rt}\lambda(t)\} - Variance\,\{e^{-rt}\gamma(t)\}$$

Here $Variance$ means the variance of the distribution of the present value of inputs $\lambda(t)$ or outputs $\gamma(t)$ over the time axis. Thus, for example, if $e^{-rt}\lambda(t)$ were equally distributed between t=-1 and $t = 0$ then $Variance\,\{e^{-rt}\lambda(t)\}$ would be $\frac{1}{12}$.

By the same mathematical formula but with a different economic interpretation we also find

$$\frac{dZ}{dr} = Variance\,\{e^{-rt}\lambda(t)\} - Variance\,\{e^{-rt}\gamma(t)\}$$

In this case $\lambda(t)$ is the labor supply of any one of the overlapping generations; and $\gamma(t)$ is the consumption stream of any one of the overlapping generations.

Now we can infer the sign of $\frac{dT}{dr}$ and $\frac{dZ}{dr}$ from the basic structure of modern economies. The division of labor characterizes the production system of the economy. In our context, this means that most investment goods are made by other people than those who use the investment goods. This is true for tangible investment goods like roads or buildings or machines; and it is true for intangible goods like the results of research and development, for example patents. But the investment goods have to be made before they can be used to produce goods in conjunction with labor inputs. In terms of the 100 percent vertically integrated hypothetical firm this means that a large fraction of labor inputs occurs long before the firm begins with the production of consumption goods. Consider the case of the hypothetical fully integrated firm that begins to produce a constant flow of consumption goods γ at time zero. Labor input $\lambda(t)$ then is positive for many (perhaps all) negative time moments, i.e. for $t < 0$. They are required to make the equipment that then is needed to make the consumption goods. But $\lambda(t)$ remains positive for $t \geq 0$. Obviously $Variance\{e^{-rt}\lambda(t)\}$ then is larger than $Variance\{e^{-rt}\gamma(t)\}$.

Now, we can view the economy at large as a collection of overlapping hypothetical firms, each of which is fully vertically integrated. Therefore the result just mentioned also applies to the economy at large. The result is

$$\frac{dT}{dr} > 0$$

We may take the Solow production function as an example. A given technique of production θ here exhibits a certain capital intensity k. The wage interest curve for the technique then is $w = y - rk$. It is a downward sloping straight line. For given y and given k the period of production T then is

$$T = -\frac{dw}{dr}\frac{1}{w} = \frac{k}{y - rk}$$

Obviously, we then obtain

$$\frac{\partial T}{\partial r} > 0$$

Now we shift from the production system to the consumption system of the economy. We look at a given generation of people after they have left their parents´ home, say at age 20. Their consumption is spread out over their life from, say, age 20 to age 80. Their supply of labor only covers part of their life above 20, say, age 20 to age 60. This induces the result that in all likelihood $Variance\{e^{-rt}\gamma(t)\}$ is larger than $Variance\{e^{-rt}\lambda(t)\}$. We then surmise that, as a rule,

$$\frac{\partial Z}{\partial r} < 0$$

In addition, if we introduce inherited wealth, the consumption flow financed by the labor supply of any given person is spread out even more.

We then assume

<u>Assumption 9: The partial derivative</u> $\frac{\partial T(r;\theta)}{\partial r}$ <u>is positive. The partial derivative</u> $\frac{\partial Z(r;\theta)}{\partial r}$ <u>is negative.</u>

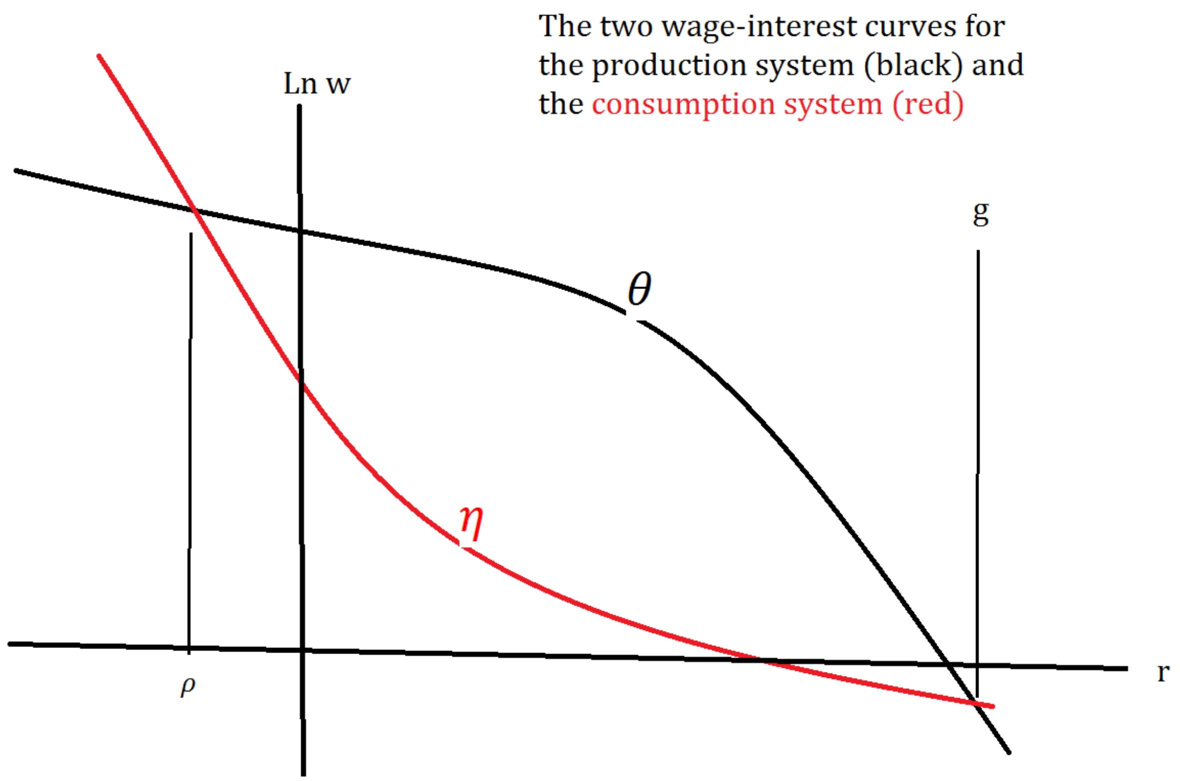
<u>Section 6. Economic Implications of the concave (convex) wage interest-curves</u>

For a given technique θ Assumption 9 implies that the natural logarithm of the wage- interest curve is a concave function: we have $\frac{\partial ln\, w}{\partial r} = \frac{\partial w}{\partial r}\frac{1}{w} = -T$; and therefore $\frac{\partial^2 \ln w}{\partial r^2} = -\frac{\partial T}{\partial r} < 0$.

On the other hand, for a given work-consumption pattern η Assumption 9 implies that the natural logarithm of the of the wage- interest curve is a convex function: $\frac{\partial ln\, \widehat{w}}{\partial r} = \frac{\partial \widehat{w}}{\partial r}\frac{1}{\widehat{w}} = -Z$; and therefore $\frac{\partial^2 \ln \widehat{w}}{\partial r^2} = -\frac{\partial Z}{\partial r} > 0$.

Let us now look at the steady state equilibrium at the natural rate of interest ρ. Given the facts of the real world we assume $\rho < g$. (But a similar analysis could be performed for $\rho > g$))

In the steady state equilibrium at the natural rate of interest the technique in use is $\theta(\rho) = \rho$. In the steady state equilibrium at the natural rate of interest the work-consumption pattern in use is $\eta(\rho) = \rho$. Now, by definition of the natural rate of interest, the government debt period $D(\rho)$ is equal to zero. Therefore $\widehat{w}\big(\varrho; \eta(\rho)\big) = w(\rho; \theta(\varrho))$. Furthermore, at $r = g$ we again have $\widehat{w}\big(g; \eta(\rho)\big) = w(g; \theta(\varrho))$. Taking account of Assumption 9 about concavity and convexity of the wage- interest curve we obtain the following graph with r on the horizontal axis and the logarithm of the wage-interest curves on the vertical axis.

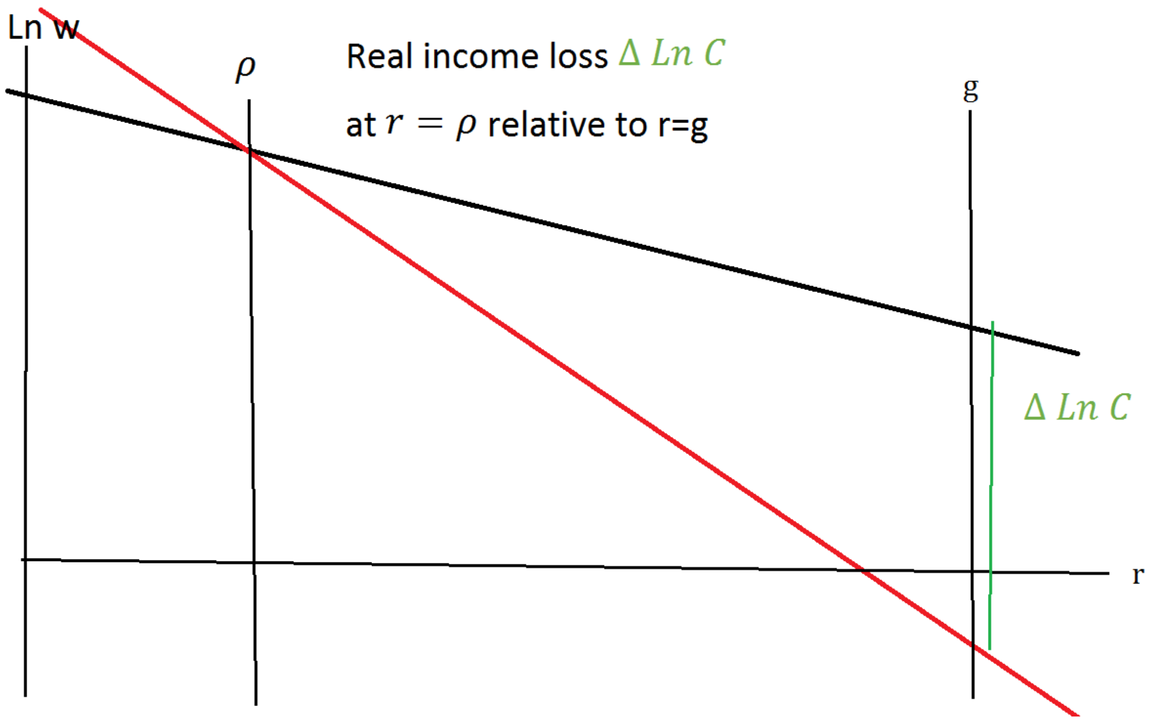


The two curves have the same value for $r = \varrho$ and for $r = g$. But the curve of the production system is concave and the curve of the consumption system is convex. Thus, for values of r between ρ and g the black curve is above the red curve. As the rate of interest rises above ρ we already know from Theorem 2 that under our Assumptions life utility rises. But this already shows that the black curve is above the red curve. Thus, our Assumption 9 appears to be redundant. Yet we have deduced it from the fact that we live in a production system characterized by the division of labor. It may be worthwhile to investigate an unrealistic world where the wage curve of the production system is convex and the wage curve of the consumption system is concave. Would such an assumption contradict one of our other assumptions? Here we do not investigate this puzzle further.

We now turn to the fact that changes in the steady state rate of interest induce substitution. By Assumptions 4 and 6 (r is an unbiased price signal in the production and in the consumption system) we can construct two envelopes. The envelope of the production system is an upper envelope of the wage interest curves of the wage- interest curves of the techniques $\theta \in Theta$. For reasons which become clear below in Chapter 3 we draw the envelope of the production system as a straight line corresponding to a stable period of production T: for any given production technique θ we know that T rises with the rate of interest. But, due to substitution there is a tendency of the period of production to decline with rising r. A stable period of production T then is the special case in which the concavity effect and the substitution effect just compensate each other.

In the consumption system signs are reversed. For any given work-consumption pattern the wage interest curve is convex. But keeping life utility $\overline{U}$ the same a rise in the rate of interest induces substitution which tends to raise the waiting period Z. We thus may again take the special case in which the convexity effect of the wage interest curve of any given work-consumption pattern is just compensated by the substitution effect induced by a change in the rate of interest. So we look at the special case in which the waiting period of the lower envelope for a given life utility level is a constant – and therefore the lower envelope of a constant utility is a straight line if we use the logarithm of the wage as a function of the rate of interest.

Now we can do welfare analysis: how much real income (in steady state consumption terms) is lost in percentage terms, if the rate of interest is not equal to the growth rate but equals the natural rate of interest? The following graph depicts the special case where the envelope period of production T and the envelope waiting period Z are straight lines, if we draw the curves in terms of the logarithm of the corresponding wage rate.



At the natural rate of interest ρ we have $\widehat{w} = w$. The red envelope describes the wage rate $\overline{w}$, which corresponds to the level of life utility $\overline{U}$ obtained at the natural rate of interest ρ in terms of the logarithm of the wage rate required to maintain the utility level. It has a slope of $-Z$. The black envelope shows the logarithm of the wage rate of the production system as a function of the rate of interest. It has a slope of $-T$.

By Theorems 1 and 2 steady state consumption and steady state life utility obtain their maximum at the rate of interest $r = g$. At this rate of interest we also have $\widehat{w} = w = c$ and $\overline{w} = \bar{c}$. Therefore at $r = g$ the distance between the black wage interest curve and the red wage-interest curve also indicates the difference in the logarithm of the maximum achievable steady state consumption $c(g; \theta(g))$ and the level of the logarithm of steady state consumption $\bar{c}$, which corresponds to the life utility $\overline{U}$ that obtains in a steady state at the natural rate of interest $r = \rho$. Let us call this difference $\Delta Ln(C)$. We then have the equation

$$\Delta Ln(C) = (g - \rho)(Z - T)$$

Or in terms of the ratio of the two consumption levels

$$\frac{\bar{c}}{c(g; \theta(g))} = e^{(\rho - g)(Z - T)}$$

As a numerical example we may assume that the natural rate of interest is $-2\%\ p.a.$ and that the steady state rate of growth, g, is $+3\%\ p.a.$ Hence the difference between the two rates would be $5\%\ p.a.$ Assume now a waiting period of $Z = 10\ years$, and a period of production of $T = 5\ years$. We then get

$$\frac{\bar{c}}{c(g;\theta(g))} = e^{(-0.05)(5)} = e^{-0.25} = 77.9\%$$

This means that with this numerical example and with the assumption that Z and T on the envelope are invariant against changes in the steady state rate of interest, the natural rate of interest induces a loss in the standard of life of 22.1% of the achievable maximum steady state standard of life.

Below, in Chapter 3, we come back to the quantification of the welfare losses due to deviations of the steady state rate of interest from the steady state rate of growth.

Section 7. The uniqueness of the natural rate of interest.

For the further theoretical and empirical analysis, it would be useful, if we could show that the natural rate of interest is unique. This would mean that there is only a single steady state rate of interest, which is compatible with a government debt ratio D equal to zero.

The assumptions, which we have introduced above, are sufficient to show uniqueness of the natural rate of interest.

First we investigate the influence of a marginal change of the steady state rate of interest on the value of the steady state capital stock, keeping its physical composition constant. We use the wage-interest-curve $w(r;\theta)$ keeping θ fixed. We have in mind a natural rate of interest $r = \rho < g$. But the following argument applies also to a natural rate of interest $r = \rho \geq g$. We have the equation for the value of the capital stock

$$v(r;\theta(\rho)) = \frac{w(r;\theta(\rho)) - c(\theta(\rho))}{g-r}$$

It is useful to provide the analogous formula for the economy at large. Here N is the number of heads at time $t = 0$. We then can write

$$V(r;\theta(\rho)) = \frac{w(r;\theta(\rho))N - C(\theta(\rho))}{g-r}$$

with $V(r;\theta(\rho)) = Nv(r;\theta(\rho))$ and $C(\theta(\rho)) = Nc(\theta(\rho))$. We can "understand" this equation better, by recognizing that in the case $r < g$ the value of the capital stock is the result of earlier activities. Indeed, for $t \leq 0$ we can write

$C_t(\theta(\rho)) = e^{gt}C_0$ and thus we get: $\int_{-\infty}^{0} e^{-rt}C_t(\theta(\rho))dt = \frac{C_0(\theta(\rho)}{g-r}$

And similarly for $\int_{-\infty}^{0} e^{-rt}w(r;\theta(\rho))N_t dt = \frac{w(r;\theta(\rho))N_0}{g-r}$.

In the case $r > g$ we integrate from time zero to time $+\infty$. In that case we get $\int_{0}^{\infty} e^{-rt}C_t(\theta(\rho))dt = \frac{C_0(\theta(\rho))}{r-g}$ and $\int_{0}^{\infty} e^{-rt}w(r;\theta(\rho))N_t dt = \frac{w(r;\theta(\rho))N_0}{r-g}$

Now we differentiate $V(r;\theta(\rho))$ partially with respect to r.

$$\frac{\partial V}{\partial r} = \frac{(g-r)N\frac{\partial w}{\partial r} + (wN - C)}{(g-r)^2} = \frac{V - NT(r;\theta(\rho))w}{g-r}$$

The partial derivative of the capital value with respect to r is known in capital theory as the "Wicksell-effect". Its sign depends on the curvature of the wage-interest curve for a given physical composition of the capital stock. Its sign is positive, if that wage-interest curve is concave. Its sign is negative, if that wage-interest curve is convex. Its sign is zero for the Solow production function, because then the wage-interest curve is a straight line. The Sraffa school in capital theory considered it very important that the Wicksell effect can be different from zero. Among other reasons it challenged the use of the Solow production function by the MIT-school in capital theory, because the Solow production function could dispense with a non-zero Wicksell effect.

In my view the Wicksell-effect can be interpreted as a signal for the size of the $Variance$ of the intertemporal distribution of labor inputs and consumption good outputs of the hypothetical fully vertically integrated firm. If the $Variance$ of the labor inputs exceeds the $Variance$ of the consumption good outputs by more than would be the case with the Solow production function then the Wicksell-effect is positive. This means that the upstream goods in the vertical ordering of the goods are made with a higher capital intensity than the downstream goods, in particular the final consumption goods.

But the result, which we want to obtain in this section, does not depend on a particular sign of the Wicksell-effect. At this point it is only important to understand the following. As long as the production technique θ and the consumption-labor pattern η do not change and as long as government debt stays at zero (we look at the natural rate of interest ϱ), the value of $\widehat{w}$ is the same as the value of w; and the value of $\hat{v}$ is the same as the value of v.

We then can write for $D = 0$ that $\frac{\partial \hat{v}}{\partial r} = \frac{\partial v}{\partial r}$.

A marginal change of the rate of interest does not only change the value of the capital stock and the wage rate. It also changes the method of production and the work-consumption pattern. Here we use Assumption 5, which reads $\frac{\partial v}{\partial \theta}\frac{1}{v} < \frac{\partial c}{\partial \theta}\frac{1}{c}$. Assumption 8 reads $\frac{\partial \bar{c}}{\partial \eta}\frac{1}{\bar{c}} < \frac{\partial \bar{v}}{\partial \eta}\frac{1}{\bar{v}}$. Now, we observe that Assumption 7 can also be expressed in the following way: for a given rate of interest the consumption per head required for some life utility level and the average amount of wealth per person move in tandem. We then can replace $\bar{c}$ in the last inequality by $\hat{c} = c$ and we can replace $\bar{v}$ by $\hat{v}$ so that we obtain the inequality

$$\frac{\partial \hat{v}}{\partial \eta}\frac{1}{\hat{v}} > \frac{\partial c}{\partial \eta}\frac{1}{c} = \frac{\partial c}{\partial \theta}\frac{1}{c}$$

We thus obtain by total differentiation with respect to r at the point $r = \rho$, so that $\hat{v} = v$, the following inequality

$$\frac{d\hat{v}}{dr}\frac{1}{\hat{v}} = \frac{\partial \hat{v}}{\partial r}\frac{1}{\hat{v}} + \frac{\partial \hat{v}}{\partial \eta}\frac{1}{\hat{v}} > \frac{\partial v}{\partial r}\frac{1}{v} + \frac{\partial c}{\partial \theta}\frac{1}{c} > \frac{\partial v}{\partial r}\frac{1}{v} + \frac{\partial v}{\partial \theta}\frac{1}{v} = \frac{dv}{dr}\frac{1}{v}$$

For $r = \varrho$ and hence $\hat{v} = v$ this inequality implies

$$\frac{d\hat{v}}{dr} > \frac{dv}{dr}$$

Now, remembering the equation $\hat{v} = v + wD$ we obtain

$$\frac{d\hat{v}}{dr} = \frac{dv}{dr} + \frac{dw}{dr}D + w\frac{dD}{dr}$$

which in the case of $r = \rho$ and hence $D = 0$ leads to

$$\frac{d\hat{v}}{dr} = \frac{dv}{dr} + w\frac{dD}{dr}$$

so that the inequality $\frac{d\hat{v}}{dr} > \frac{dv}{dr}$ implies, together with $w > 0$, that

$$\frac{dD}{dr} > 0 \text{ at } r = \rho$$

But this implies uniqueness of a steady state interest rate with zero public debt.

Theorem 4: With Assumptions 1-8 there is at most one steady state rate of interest, which is compatible with a zero public debt period D. If ρ is that steady state rate of interest then $D(r) > 0$ for $r > \rho$ and $D(r) < 0$ for $r < \rho$.

Proof: Assume the contrary. We then can find at least two distinct natural rates of interest ϱ_1 and ϱ_2 with $\varrho_1 < \varrho_2$. Because of Assumption 3 the functions $w(r;\theta)$, $\overline{w}(r;\eta(\overline{U}$,r)); $v(r;\theta)$ and $\bar{v}(r;\eta(\overline{U};r)$ are continuously differentiable functions of r and θ. We then can also infer that $\widehat{w}(r;\eta(r))$ and $\hat{v}(r;\eta(r))$ are continuously differentiable. Because of $\frac{dD}{dr} > 0$ at $r = \rho_1$ we have $D(r) > 0$ for $r > \rho_1$ for r sufficiently close to ρ_1. But then because of the continuity of the function $D(r)$ there exists ρ_2, such that $D(r) > 0$ for $\rho_1 < r < \rho_2$ and $D(\rho_2) = 0$. But then $\frac{dD}{dr} \leq 0$ at $r = \rho_2$, in contradiction to the inequality derived above: $\frac{dD}{dr} > 0$. This shows that such ρ_2 cannot exist. But then, because of $\frac{dD}{dr} > 0$ at $r = \rho_1 \equiv \rho$ we have $D(r) > 0$ for $r > \rho$ and $D(r) < 0$ for $r < \rho$. QED.

Section 8. The Wage-Interest Curve and the Consumption-Growth Curve

For a given technique θ let $w(r;\theta)$ be the wage-interest curve. We rewrite our equation (1), by adding g as an argument

$$c(g;r;\theta) + gv(g;r;\theta) = w(g;r;\theta) + rv(g;r;\theta)$$

If we can understand the steady state economy to be an appropriate collection of overlapping hypothetical 100 %-vertically integrated firms, we can conclude that $w(g;r;\theta)$ for given θ does not depend on g. On the other hand, we have argued that by an appropriate choice of the price level that for given θ $c(g;r;\theta)$ does not depend on r. We then can write

$$c(g;\theta) = w(r;\theta) \text{ for } r = g$$

This means the curve describing the wage-interest curve also describes the consumption-growth curve, provided the same technique prevails. This is the basic duality relation of steady state capital theory.

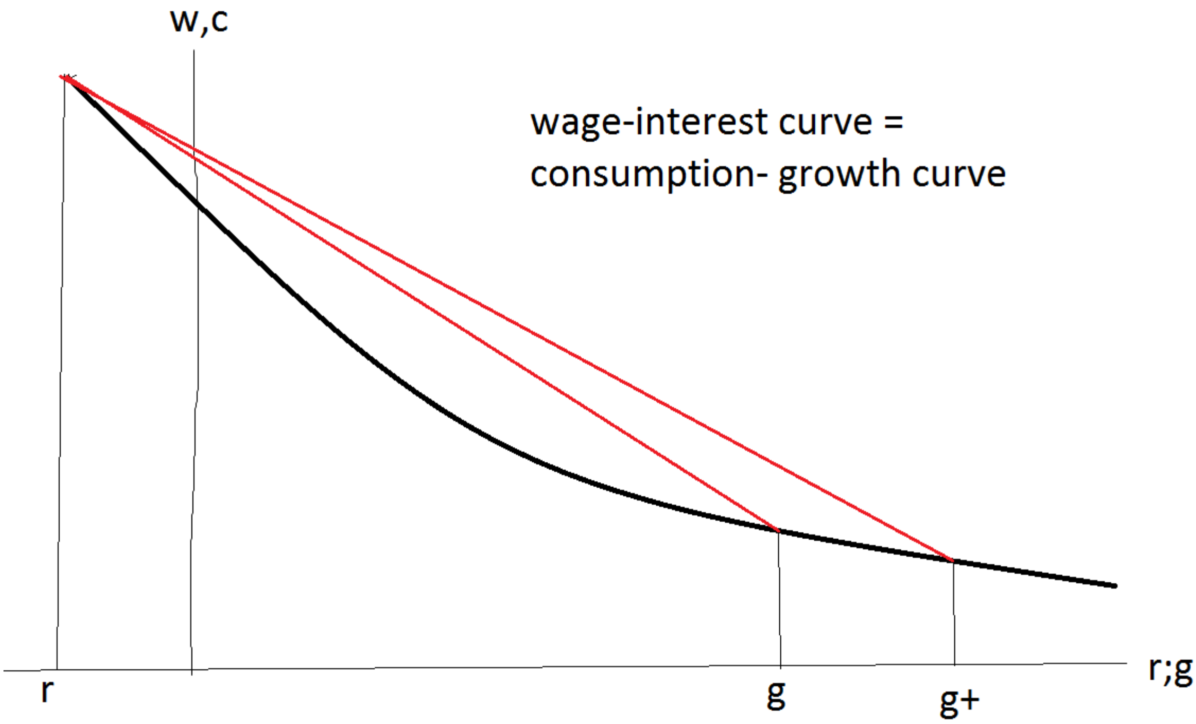
We may give this insight the name of

Theorem 5: Assumption: the steady state economy can be seen as a collection of overlapping hypothetical fully vertically integrated firms with the result that $w(r;\theta)$ is not influenced by the growth rate of the system. Then the basic duality relation of steady state capital theory holds: for any given technique θ the wage-interest curve is the same as the consumption-growth tradeoff curve applicable with the same technique.

As an example, we can use the basic duality relation and the Wicksell- effect for the following policy issue. One frequently hears the following proposal to solve the conundrum of the savings glut: policy should change so that private investment is encouraged. Thereby, one hopes, one also can lift the equilibrium rate of interest, without having to incur additional public debt.

But, if one, for example, uses a Solow-production function a higher rate of interest reduces the capital intensity of production. And, also our Assumption 5 (the "law of demand") predicts a lower capital-output ratio from a higher rate of interest. Therefore, unless one wants to construct pyramids, like in ancient Egypt, the stimulus for investment must be accompanied by a higher steady state growth rate of the economy. For the welfare of people it may be fruitful to change things in such a way that firms are encouraged to invest more. But only, if the economy thereby can enjoy more growth.

However, it is not a foregone conclusion that additional growth will raise the demand for capital in the steady state. Indeed, we know for the Solow-production function that the capital intensity k is not affected by a different rate of growth, as long as the rate of interest remains the same. More generally, if the wage-interest curve for a given technique is convex then a higher growth rate of the system reduces the demand for capital in the steady state. This result is just the duality mirror image of a negative or, "neoclassical" Wicksell effect. The figure depicts a convex wage-interest curve. Keeping the rate of interest fixed, we compare two different growth rates g and $g+$ with $g+> g$. The slopes of the two corresponding red lines provide the value of capital $v(g;r;\theta)$. Here we see that the higher growth rate comes with a lower capital requirement. The same result can be obtained algebraically.



It is, therefore, not at all clear that a growth stimulating policy provides an answer to the savings glut problem, if then it is a problem.

<u>Section 9. Synchronization. Explaining the representative household.</u> We work in a model of overlapping generations. Indeed, working in continuous time, at any one point in time there co-exist many overlapping generations. Even if persons are of the same "type" concerning their abilities, their life expectancy and their preferences - they differ by age of birth. What does then the concept of the representative household mean? The basic idea is the principle of synchronization. It is an old idea, going back to Marx, Alfred Marshall, Böhm-Bawerk, John Bates Clark and others. In our context it means the following.

It is easily explained in the setting of a stationary economy. Assume citizens are all of the same "type" concerning their tastes and opportunities. But they have different dates of birth. There is no technical progress. Total population is stationary. It is then the case that the distribution of consumption across the different age cohorts at any given moment of time is just a mirror image of the intertemporal distribution of consumption of any given cohort. The same is the case for the supply of labor.

We then can use the intertemporal pattern η of consumption and labor supply to derive the present quantity of consumption and labor employed. In value terms: if the rate of interest is zero then the value of present total consumption equals the present value of life consumption of any one of the cohorts. And the value of the present wage bill equals the present value of life wage earnings accruing to any one of the cohorts. This mirror image of the intertemporal pattern in today´s inter-cohort pattern of distribution of present labor and consumption is what we might call <u>perfect synchronization</u>. Given that all cohorts are alike, no distributional issues arise. The welfare analysis of the distribution of work and of consumption among different cohorts then simply boils down to the life utility analysis of any one of these cohorts. The figure of the representative household then is justified by the assumption of perfect synchronization.

<u>Section 10.</u> <u>Generalized Synchronization.</u> On the production side our analysis of optimal public debt operates with a "wage"-interest frontier. It basically is a highly aggregated <u>indirect</u> production function. And on the household side we work with a highly aggregated <u>indirect</u> utility function. We do not have to specify the thousands of different commodities, which are being produced and consumed in the economy. Only the period of production *T* and the waiting period *Z* are relevant – and, of course, the public debt period *D* itself.

We thus can generalize our analysis far beyond the model of the stationary economy discussed in the preceding section. In a first step I discuss population growth. If the population grows at a rate $g > 0$ per annum and if the available technology is constant through time then perfect synchronization corresponds to the golden rule path with an interest rate equal to *g.* Then the present distribution of consumption across different cohorts is a mirror, one to one, of the present value of the intertemporal distribution of consumption of any given cohort. Again, we can apply the figure of the representative household, due to perfect synchronization.

But we can go beyond the assumption of a constant technology. Traditional growth theory explains the growing standard of life mainly by technical progress. Using the figure of the representative consumer, we can generalize our approach, to include changes through time like, for example, technical progress. In addition, certain forms of demographic change are compatible with our approach. Take technical progress first. As long as it keeps the period of production and the waiting period the same, the formula for the optimal level of public debt remains intact. In Section 5 above I describe how one computes the period of production. For this purpose I describe a hypothetical firm which is fully vertically integrated. Its only input bought from outside is labor. Its only output sold on the market consists of consumption goods. Within a steady state economy the production system can be decomposed into a set of such fully vertically integrated hypothetical firms. One then can show that on the golden rule path the capital/consumption ratio of the economy equals this period of production. Borrowing from the well- known concept of Harrod-neutral technical progress we can replace simple labor time by labor time in efficiency units. If wages per natural time unit rise in such a way that the wage of an efficiency unit remains unchanged we can apply the same approach of a hypothetical fully vertically integrated firm to determine the period of production. Again, on the golden rule path this period of production equals the capital/consumption ratio.

On the golden rule path, keeping the period of production the same across technical changes, many things may change through time. New production methods may replace the old ones. New products may replace old products. Relative prices of different consumption goods will change. The relative weight of different production sectors in the economy may change. In the hypothetical fully integrated firm the particular time pattern of (efficiency) labor inputs and consumption good outputs may change, as long as the period of production does not change.

Also the requirements for training and thus human capital may change. This may imply that the relative weight of the educational sector rises and that people on average enter the labor force at a higher age. Thus, in the economy the relative weight of the human capital content of produced goods may rise. The formula for the optimal public debt level $D = Z - T$ remains unaffected by this as long as the period of production and the waiting period do not change.

Similar considerations apply for the waiting period *Z*. For a given waiting period along the golden rule path many things may change for consumers. Their consumption basket changes. The time pattern of their use of time for work and leisure may change substantially even without affecting *Z*. The standard model is even applicable, if we no longer assume that the utility function of the representative citizen is fixed. I believe we can cope with endogenously changing preferences as long as preferences are “adaptive”. On my theory of adaptive preferences cf. my paper Weizsäcker (2013).

The dynamics of change and progress may also have an impact on positive and negative externalities, on the forms and the intensity of competition.

Given that we deal with a wide class of changes in technology and thus with a rising trend in the standard of living, we do have an issue of intergenerational distribution. Younger cohorts are likely to be better off than older cohorts. In the standard model I ignore this issue. Also, given that I assume only one "type" I ignore intra-generational distribution issues.

I conclude that the standard model of a given period of production and a given waiting period along the golden rule path is quite general. Of course, it is true that there are many potential causes generating changes in *T* or *Z*. Moreover the unbiasedness of the price signal *r* may be at odds with reality. I discuss some of the deviations from the standard model in later chapters.

Chapter 3. The Coefficient of Intertemporal Substitution

Section 1. Introduction. This paper on the capital theory of the steady state provides theoretical background for policy related work to cope with the challenge of a negative natural rate of interest. There is a theoretical challenge: the CES production function, which was introduced by Arrow, Chenery, Minhas and Solow in 1961. It became popular in empirical and forecasting work because it allowed working with an empirically measured elasticity of substitution between labor and capital in forecasting exercises, which had to deal with future capital intensities far outside the range that was observed in the past. By assuming that the elasticity of substitution between labor and capital is the same in the future as was estimated for the past one could provide forecasts of real domestic product, investment levels, consumption levels etc. However, in the CES production function the marginal productivity of capital remains positive for all levels of the capital intensity. This is a challenge for a theory dealing with the negative natural rate of interest.

Here I propose a substitute for the elasticity of substitution between capital and labor. It is the coefficient of intertemporal substitution. For a steady state analysis it has substantial advantages, as will be shown in this Chapter 3. In particular, it can cope with a negative rate of interest, i.e. with a negative marginal productivity of capital – even if we work with the assumption that the coefficient of intertemporal substitution is a constant across different capital intensities. Moreover, it is better suited for a meta-model like ours. We are interested in quantifying the welfare losses, which are induced by a divergence between the rate of interest r and the steady state rate of growth g. Here the coefficient of intertemporal substitution helps. In addition, using the concepts of the period of production T and the "waiting period" Z, it allows some substantial formal symmetry between the production system and the consumption system of the economy. Furthermore, by assuming that the coefficient of intertemporal substitution is a constant, we get access to the size of that constant in the production system, by using the well-known "Kaldor fact" that the capital output ratio has no trend through time.

Section 2. The Law of Intertemporal Substitution in the Production System

In this section, I prove a theorem, which I call the "Law of Intertemporal Substitution". Here I explain this name. Production is an activity, which takes time. On average inputs have to be available before outputs accrue. Production then transforms goods of today into goods, which are available at future dates. Böhm-Bawerk´s average period of production attempts to quantify how far away in time outputs are from inputs, on average. Thus, a change in the production mode, which lengthens or shortens the average period of production is an intertemporal substitution. A longer period of production replaces earlier outputs by later outputs. For given inputs, people have to wait for a longer time to obtain the fruits of the production process. As the following Theorem 6A is about changes in the period of production, I call it the Law of Intertemporal Substitution, Part1.

Similar considerations apply to the household sector or, as we also call it: the "consumption system". A rise in the waiting period means that people wait for a longer time until they consume the fruits of their labor. They replace consumer goods by other consumer goods, which accrue later. Again, this means that some intertemporal substitution takes place. As the following Theorem 6B, derived in Section 3, is about changes in the waiting period, I call it the Law of Intertemporal Substitution, Part 2.

In the following we do not need Assumption 5 (the "law of demand" in the production system).

Here then comes

Theorem 6A (Law of intertemporal Substitution, Part 1): Under Assumptions 1,2,3 and 4 the following holds: Keeping the weighting system r the same, a small rise in the value of θ reduces the period of production: in a formula:

$$T_2(r;\theta(r)) \leq 0 \qquad (5)$$

Proof: In section 3 of Chapter 2 above we already have discussed the first total derivative of the "wage" function with respect to r:

$\frac{dw(r;\theta(r))}{dr} = w_1 + w_2 = -T\big(r;\theta(r)\big)w\big(r;\theta(r)\big) + w_2(r;\theta(r))$.

I now discuss the second total derivative of w with respect to r. We obtain

$$\frac{d^2w(r;\theta(r))}{dr^2} = w_{11} + w_{12} + w_{21} + w_{22}$$

We can specify this equation by means of the following characteristics. Because of Assumption 4 (Unbiased price signal r in the production system) we know that $\frac{d^2w(r;\theta(r))}{dr^2} \geq w_{11}(r;\theta(r))$ (envelope theorem) and we know that $w_2(r;\theta(r)) = 0$ and $w_{22}(r;\theta(r) \leq 0$. Furthermore $w_{21} = w_{12} = -wT_2 - w_2T = -wT_2$. Thus $\frac{d^2w(r;\theta(r))}{dr^2} - w_{11}\big(r;\theta(r)\big) - w_{22}(r;\theta(r) \geq 0$; or, which is the same, $2w_{12}\big(r;\theta(r)\big) = -2wT_2 \geq 0$. This implies $T_2 \leq 0$. QED

The economic meaning of this inequality $T_2 \leq 0$ is this: Keeping the "weighting system" r the same, the physical change of the production system upon a "small" rise of the rate of interest leads to a shorter period of production. In the language of Böhm-Bawerk: as the roundaboutness of production becomes more expensive the degree of roundaboutness of production declines. This is the first part of the Law of Intertemporal Substitution.

It is useful to define a measure for the power of the intertemporal substitution. The traditional elasticity of substitution between two inputs is dimensionless, hence does not depend on the choice of units, which measure the quantities of the two inputs. Following this lead, I also define a coefficient of intertemporal substitution, which is dimensionless. I give it the symbol ψ.

Definition: The Coefficient of Intertemporal Substitution ψ: It is defined by this equation $\psi = \frac{\partial \frac{1}{T(r;\theta(r)}}{\partial\theta} = \frac{d\frac{1}{T}}{dT}\frac{\partial T(r;\theta(r))}{\partial\theta} = -\frac{1}{T^2}T_2(r;\theta(r))$.

Or the other way round: $T_2 = -T^2\psi$. As we have shown that $T_2(r;\theta(r))$ is negative, or at most zero, we know that ψ is nonnegative. It is dimensionless, because $1/T$ and r and $\theta(r)$ have the same dimension.

Section 3. The Law of Intertemporal Substitution in the Consumption System

In the following we look at the "compensated demand function" or "compensated supply function"; keeping life utility the same we look at the reaction of labor supply and consumption goods demand upon a small rise in the rate of interest r. We fix the level of life utility U at some level $\overline{U}$. For convenience of notation we skip writing down the constant $\overline{U}$. The labor-consumption pattern then is a function of r and of $\overline{U}$. We then can write $\eta = \eta(r)$. And we give $\eta(r)$ the "name" r; i.e. $\eta(r) = r$.

Note that in the following we do not need Assumption 8 (The "law of demand" and the "law of supply" in the consumption system).

Theorem 6B: (Law of intertemporal Substitution, Part 2): Under Assumptions 1,3,6,and 7 the following holds: Keeping the weighting system r the same and keeping life time utility the same, a small rise in the value of η raises the value of the waiting period: in a formula:

$$Z_2(r;\eta(r)) \geq 0 \qquad (6)$$

Proof: As discussed earlier Assumption 6 (UBPS2) means that the representative household maximizes life time utility under the given budget constraint $\overline{w}$ and r. This is equivalent to the property that for given r the "wage" $\overline{w}(r;\eta(r))$ is the minimum of all w for financing some η such that $U(\eta) \geq \overline{U}(\eta(r))$. The total first derivative of $\overline{w}$ with respect to r then is $\frac{d\overline{w}(r;\eta(r))}{dr} = \overline{w}_1(r;\eta(r))+\overline{w}_2(r;\eta(r))$. We then know from Chapter 2, section 4, that $\overline{w}_1(r;\eta(r)) = -Z(r;\eta(r))\overline{w}(r;\eta(r))$. Moreover UBPS2 implies $\overline{w}_2(r;\eta(r)) = 0$. The second total derivative then is

$$\frac{d^2\overline{w}(r;\eta(r)}{dr^2} = \overline{w}_{11}(r;\eta(r)) + \overline{w}_{12}(r;\eta(r)) +$$

$$+\overline{w}_{21}(r;\eta(r)) + \overline{w}_{22}(r;\eta(r))$$

Furthermore $\overline{w}_{12}(r;\eta(r)) = \overline{w}_{21}(r;\eta(r)) =$

$$-Z_2(r;\eta(r)\overline{w}(\mathrm{r};\eta(\mathrm{r})) - \mathrm{Z}\overline{w}_2(\mathrm{r};\eta(\mathrm{r})) = -Z_2(r;\eta(r)\overline{w}(\mathrm{r};\eta(\mathrm{r})).$$

So we get

$$\frac{d^2\overline{w}(r;\eta(r))}{dr^2} = \overline{w}_{11}(r;\eta(r)) -$$

$$-2Z_2(r;\eta(r)\overline{w}(r;\eta(r)) + \overline{w}_{22}(r;\eta(r))$$

But because of UBPS2 we know that $\overline{w}_{22}\big(r;\eta(r)\big) \geq 0$ and that $\frac{d^2\overline{w}(r;\eta(r))}{dr^2} \leq \overline{w}_{11}\big(r;\eta(r)\big)$ (envelope theorem). The following inequality follows

$\frac{d^2\overline{w}(r;\eta(r))}{dr^2} - \overline{w}_{11}\big(r;\eta(r)\big) - \overline{w}_{22}\big(r;\eta(U;r)\big) \leq 0$, or, which is the same,

$-2Z_2(r;\eta(r)\overline{w}(r;\eta(r)) \leq 0$ i.e. $Z_2(r;\eta(r)) \geq 0$. QED.

Given that we know the sign of the intertemporal substitution effect we can use it to define a second coefficient of substitution, which we call γ. As in the case of ψ we want it to be dimensionless. Thus,

Definition: The Coefficient of Intertemporal Substitution γ: It is defined by the following equation: $\gamma = -\frac{\partial \frac{1}{Z}}{\partial \eta} = -\frac{d\frac{1}{Z}}{dZ}Z_2 = \frac{Z_2}{Z^2}$.

This is equivalent to $Z_2 = \gamma Z^2(r;\eta(U;r))$.

As we shall see below in Sections 4 and 5, the coefficients of intertemporal substitution γ and ψ are useful in explaining the shortfall of "real income" from its maximum, if the rate of interest diverges from its optimal value.

Section 4. Quantifying Welfare Losses for Suboptimal Interest Rates in the Meta-Model: The Production System.

It is of interest to know the loss function in case of a suboptimal rate of interest. Our approach allows us to find a simple loss formula for "small" deviations from the optimal rate. It is an approximation for our meta-model. The approximation is a second order Taylor expansion around the optimal rate $r = g$. It can best be written down by using the two coefficients of intertemporal substitution ψ and γ, which we introduced above in Section 2 and Section 3. For any specific model, like for example the Solow production function, one can check how well the approximation works.

I begin with the loss function for the production system.

We know that $w(g;g) \geq w(g;\theta(r))$ for all $\theta(r) \in Theta$. I now develop an approximation for the welfare loss $w(g;g) - w(g;\theta(r))$. It is a second order Taylor approximation. Our approximation formula then reads

$$w(g;\theta) = w(g;g) + w_2(g;g)(\theta - g) + w_{22}(g;g)\frac{1}{2}(\theta - g)^2 = w(g;g) + w_{22}(g;g)\frac{1}{2}(\theta - g)^2$$

because $w_2(g;g)$ is of course zero. I now compute the value $w_{22}(g;g) = \frac{\partial^2 w}{\partial\theta^2}(g;g)$.

<u>Theorem 7A: The second order Taylor approximation of the proportional real income loss relative to the optimum on the production side is $\psi T^2(r-g)^2/2$, with ψ the coefficient of intertemporal substitution in the production system and T the "period of production", both at the optimum.</u>

<u>Proof:</u> The second order Taylor approximation is equivalent to the replacement of the true function $w(r;\theta)$ by a quadratic function in r and θ. We then write

$$w(r;\theta) = w^* + a_1(r-g) + a_2(\theta-g) + a_3(r-g)^2 + a_4(r-g)(\theta-g) + a_5(\theta-g)^2$$

Here then $w^* \equiv w(g;g)$. Because of UBPS1 we know for $\theta = r$ that the partial derivative with respect to θ is equal to zero. Hence for every value of r

$$\frac{\partial w}{\partial\theta}_{\theta=r} = a_2 + a_4(r-g) + 2a_5(\theta-g) = 0$$

which implies $a_2 = 0$ and $a_4 = -2a_5$.

Moreover we have $\frac{\partial^2 w}{\partial\theta^2} = 2a_5 = -a_4$. For $T = -\frac{\partial w/\partial r}{w}$ we obtain

$$T = -\frac{a_1 + 2a_3(r-g) + a_4(\theta-g)}{w^*}$$

Keeping in mind $\frac{\partial w}{\partial\theta}_{\theta=r} = 0$ (so that the numerator´s partial derivative is zero) it follows

$$\frac{\partial T}{\partial\theta} = -\frac{a_4}{w} = \frac{1}{w}\frac{\partial^2 w}{\partial\theta^2} \text{ or, for } r-g=0 \text{ and } \theta-g=0$$

$$w_{22} \equiv \frac{\partial^2 w}{\partial\theta^2} = w(g;g)\frac{\partial T}{\partial\theta} = -w^*\psi T^2$$

We then have

$$w(g;\theta) = w^* + w_{22}(g;g)\tfrac{1}{2}\theta^2 = w^*\left[1 - \tfrac{1}{2}\psi T^2(\theta-g)^2\right]$$

<u>QED</u>

Because we apply a second order Taylor approximation the welfare loss obviously must rise with the square of the deviation from the optimal rate of interest. Moreover the proportional welfare loss is also in proportion to the coefficient of intertemporal substitution ψ.

Section 5. Quantifying Welfare Losses for Suboptimal Interest Rates: The Consumption System.

In the following U stands for $U(\eta(g))$. This is the life-time utility obtained in case of the optimal rate $r = g$. As discussed above, we can form the function $\bar{w}(g;\eta(U,r))$ and the expression $\bar{w}(g;\eta(U,g))$. These are the wage rates required for the financing of labor-consumption patterns $\eta(U,r)$ and $\eta(U,g)$. UBPS2 implies that $\bar{w}(g;\eta(U,r)) \geq \bar{w}(g;\eta(U,g))$ for all values of $r \in R$.

Note that for any work-consumption pattern η at the steady state rate of interest r equal to the steady state rate of growth g the "wage rate" $\bar{w}$ equals the corresponding consumption per head $\bar{c}$. So we also have

$$\bar{c}(\eta(U,r)) \geq \bar{c}(\eta(U,g))$$

I now develop an approximation for the difference $\bar{w}\big(g;\eta(U,r)\big) - \bar{w}(g;\eta(U,g))$. It is a second order Taylor approximation. Thus it is quite accurate for "small" deviations of r from its optimal value . I thus compute the value $\bar{w}_{22}(g;g) = \frac{\partial^2 \hat{w}}{\partial \eta^2}(g;g)$. Our approximate formula then reads

$$\bar{w}\big(g;\eta(U,r)\big) = \bar{w}\big(g;\eta(U,g)\big) + \bar{w}_2(g;g)(\eta - g) + \bar{w}_{22}(g;g)\frac{1}{2}(\eta - g)^2$$
$$= \bar{w}\big(g;\eta(U,g)\big) + \bar{w}_{22}(g;g)\frac{1}{2}(\eta - g)^2$$

because, of course, $\bar{w}_2(g;g) = 0$.

Theorem 7B: <u>The second order Taylor approximation of the proportional real income loss relative to the optimum on the consumption side is $\gamma Z^2(r-g)^2/2$, with γ the coefficient of intertemporal substitution in the consumption system and Z the "waiting period", both at the optimum.</u>

Proof*:* The second order Taylor approximation is equivalent to the replacement of the true function $\bar{w}(r;\eta)$ by a quadratic function of r and θ. We then write

$$\bar{w}(r;\eta) = \bar{w}(g;g) + b_1(r-g) + b_2(\eta - g) + b_3(r-g)^2 + b_4(r-g)(\eta - g) + b_{5(}\eta - g)^2$$

Because of UBPS2 we know for $\eta = r$ that the partial derivative with respect to η is equal to zero. Hence for every value of r we have

$$\frac{\partial \bar{w}}{\partial \eta}_{\eta=r} = b_2 + b_4(r-g) + 2b_5(\eta - g) = 0$$

which implies $b_2 = 0$ and $b_4 = -2b_5$.

Moreover we can compute $\frac{\partial^2 \overline{w}}{\partial \eta^2} = 2b_5 = -b_4$. For Z$= -\frac{\partial \overline{w}/\partial r}{\overline{w}}$ we obtain

$$Z = -\frac{b_1 + 2b_3(r-g) + b_4(\eta - g)}{\overline{w}}$$

From this – keeping in mind $\frac{\partial \overline{w}}{\partial \eta}_{\eta = r} = 0$ - follows

$$\frac{\partial Z}{\partial \eta} = -\frac{b_4}{\overline{w}} = \frac{1}{\overline{w}}\frac{\partial^2 \overline{w}}{\partial \eta^2} \text{ or, for } r = g \text{ and } \eta = g$$

$$\overline{w}_{22}(g;g) \equiv \frac{\partial^2 \overline{w}}{\partial \eta^2} = \overline{w}\frac{\partial Z}{\partial \eta} = \overline{w}(g;g)\gamma Z^2$$

We then have

$\overline{w}(g;\eta) = \overline{w}(g;g) + \frac{1}{2}\overline{w}_{22}(g;g)(\eta - g)^2$=

$\overline{w}(g;g)\left[1 + \frac{1}{2}\gamma Z^2(\eta - g)^2\right]$=$\overline{w}(g;g)\left[1 + \frac{1}{2}\gamma Z^2(r - g)^2\right]$

Thus,

$$\bar{c}(\eta(r)) = \bar{c}(\eta(g))\left[1 + \frac{1}{2}\gamma Z^2(r-g)^2\right]$$

QED.

The proportional welfare loss on the consumption side due to a "wrong" rate of interest then amounts to $\frac{1}{2}\gamma Z^2(r-g)^2$. Because we apply a second degree Taylor approximation the welfare loss obviously must rise with the square of the deviation from the optimal rate of interest. Moreover the proportional welfare loss is also in proportion to the coefficient of intertemporal substitution γ.

Section 6. The total loss in real income due to deviation of r from its optimum value g

We now can combine the loss in real income on the production side and the consumption side. The reference point is life utility U obtainable at $r = g$. It corresponds to the wage rate at that point:

$$w^* \equiv w(g;g) = \widehat{w}(g;g) = \overline{w}\big(g;\eta(g;U)\big).$$

Now, according to a second order Taylor approximation Theorem 6A tells us the following: for a the production technique $\theta(r) = r$, the wage rate that can be paid at an interest rate $r = g$ is given by

$$w(g;\theta) = w^* + w_{22}(g;g)\frac{1}{2}\theta^2 = w^*\left[1 - \frac{1}{2}\psi T^2(\theta - g)^2\right]$$

And thus, also

$$c(\theta) = w^*\left[1 - \frac{1}{2}\psi T^2(\theta - g)^2\right]$$

On the other hand, the consumption per head required to obtain the utility level U by means of the work-consumption pattern $\eta(r;U)$ can be estimated by a second order Taylor approximation, using Theorem 6B to be

$$\bar{c}(\eta(r;U) = \bar{w}(\mathrm{g};\eta(r)) = \bar{w}(g;g)\left[1 + \tfrac{1}{2}\gamma Z^2(r - g)^2\right] =$$
$$c(g;g)\left[1 + \tfrac{1}{2}\gamma Z^2(r - g)^2\right]$$

So at r the gap between the consumption required to obtain live utility U and the actually obtainable consumption $c(\theta(r))$ is given by

$$\begin{aligned}\bar{c}(\eta(r;U)) - c\big(\theta(r)\big) &= c(g)\left\{\left[1 + \frac{1}{2}\gamma Z^2(\eta - g)^2\right] - \left[1 - \frac{1}{2}\psi T^2(\theta - g)^2\right]\right\}\\ &= c(g)\frac{1}{2}\{\psi T^2(r - g)^2 + \gamma Z^2(r - g)^2\}\\ &= c(g)\frac{1}{2}(\psi T^2 + \gamma Z^2)(r - g)^2\end{aligned}$$

In other words: We can write as

<u>Theorem 7: The total proportionate loss of welfare due to a steady state rate of interest different from the rate of growth Ω, is estimated by a second order Taylor approximation to be</u>

$$\Omega = \frac{1}{2}(\psi T^2 + \gamma Z^2)(r - g)^2$$

This being a second order Taylor approximation the proportionate loss rises with the square of $r - g$.

Note that Ω is invariant against changes in the time unit, because $T(r - g)$ and $Z(r - g)$ are invariant against a change in the time unit, say from “year” to “month” – and ψ as well as γ are dimension free.

Section 7 Calibrating the coefficient of intertemporal substitution ψ for the production system, using the Solow model.

In theoretical and empirical macro-economics the Solow production function is the "workhorse" that works for the production system of the economy. It is therefore useful to connect our coefficient of intertemporal substitution with the Solow model. However, one of the reasons I have introduced this coefficient of intertemporal substitution is that we can use it much more widely. It is not restricted to the Solow model.

I describe a Solow model applicable for a given moment of time with the technology available at that time. So, for simplicity of notation, we can ignore technical progress. I also assume constant returns to scale on the macro level. In that case we can write the production function in the following way

$$f = f(k);\ f'(k) \geq 0 \text{ for } 0 \leq k \leq \bar{k} \leq \infty;\ f''(k) < 0;\ k \geq 0; \text{ if } \bar{k} = \infty, \text{ then } \lim_{k \to \infty} f'(k) = 0$$

Here f is net output per unit of labor and k is the capital-labor ratio. Its value $\bar{k}$ maximizes $f(k)$. If factor prices w for labor and r (the rate of interest) for capital correspond to their respective marginal productivities we obtain the following equations

$$r = f'(k) \text{ and } w = f - f'(k)k$$

In our meta-model of the preceding chapter the production system was called θ; and θ was "named" in such a way that $\theta(r) = r$. In the special case of the Solow model this convention to "name" the different production systems then implies the following function for the capital-labor ratio k. Because we assume that the second derivative $f''(k)$ is strictly negative the function $f'(k)$ is invertible. We thus have a function

$$k = h(f') = h(r) = h(\theta)$$

Here we observe $h'(f') = \frac{dk}{dr} = 1/\frac{dr}{dk} = \frac{1}{f''}$. This means: $''(k)h'(f') = 1$. Of course the combined mapping $h(f'(k))$ maps any given k into itself. Thus the slope of the combined mapping must be equal to unity.

We then can apply the formalism developed in the preceding chapter for the meta-model. However, it is convenient in the case of the Solow model to work with "real" values, rather than "nominal" values. Yet, we still can write a function for w in the form $w = w(r; \theta)$. And thus, again we can form the first partial derivatives of this function $w_1 = \frac{\partial w}{\partial r}$ and $w_2 = \frac{\partial w}{\partial \theta}$. As there is a one-to-one relation between θ and k we can identify w_1 with a marginal change in the wage rate upon a marginal change of the rate of interest, keeping the capital-labor ratio constant. Writing the wage rate in the form $w = f(k) - rk$ we then obtain

$$w_1 = -k.$$

As in the meta-model of the preceding chapters we can derive the following property of w_2. If we evaluate w_2 at the point $\theta = r$ we have

$$\begin{aligned} w_2 = \frac{dw}{dr} - w_1 &= \left(f'(k) - f'(k) - kf''(k)\right)\frac{dk}{dr} + k \\ &= \quad k\left(1 - f''(k)h'(\theta)\right) = 0 \end{aligned}$$

We remember that in the meta-model UBPS1 implied $w_2 = 0$ at $\theta = r$. And, of course, the assumption $r = f'(k)$ is in line with the assumption UBPS1.

The Solow model describes the economy as if it were a one commodity economy. As we discussed above, application of the Böhm-Bawerk/ Hicks period of production to the Solow model leads to the result

$$T = -\frac{w_1}{w} = \frac{k}{f - rk}$$

Now I derive the Solow production function $f(k)$, which corresponds to the assumption that the coefficient of intertemporal substitution ψ is a constant. So, in the following $\psi = \frac{\partial \frac{1}{T}}{\partial \theta}$ is a constant. We have $\frac{\partial T}{\partial \theta} = -\psi T^2$. Furthermore, from $T = -\frac{w_1}{w} = \frac{k}{f-rk}$ we derive $\frac{\partial T}{\partial r} = \frac{(-k)(-k)}{(f-rk)^2} = \frac{k^2}{(f-rk)^2} = T^2$. It follows

$$\frac{dT}{dr} = \frac{\partial T}{\partial r} + \frac{\partial T}{\partial \theta} = (1 - \psi)T^2$$

For $\psi \neq 1$ integration of this differential equation for T as a function of r leads to

$$T = \frac{1}{(\psi - 1)}\frac{1}{(r + a)}$$

Here a is a constant of integration. Using the equation $r = f'(k)$ and the equation $T = \frac{k}{f(k) - kf'(k)}$ we then can write

$$\frac{k}{f-kf'(k)}=\frac{1}{(\psi-1)}\frac{1}{(f'(k)+a)}$$

or, which is the same

$$f'(k)=\frac{1}{\psi}\frac{f(k)}{k}-\frac{\psi-1}{\psi}a$$

This is a linear differential equation for f as a function of k. Integration leads to

$$f(k)=Bk^{\frac{1}{\psi}}-ak$$

B then is another constant of integration

We now distinguish between case 1: $\psi>1$ and case 2: $0<\psi<1$.

Case 1: From $f''<0$ and: if $\bar{k}=\infty$, then $\lim_{k\to\infty}f'(k)=0$) we have to assume that the constant of integration B is positive and that the constant of integration a is nonnegative. We then have a Cobb-Douglas function minus a term which is in proportion to k. The latter term may be zero. If a is positive then $f(k)$ reaches a maximum at a finite value of k.

We replace the constant of integration B by a term which is a function of $\bar{k}$, the value of k at which $f(k)$ reaches its maximum. The production function then reads:

$$f(k)=a\left\{\psi\bar{k}\left(\frac{k}{\bar{k}}\right)^{\frac{1}{\psi}}-k\right\}$$

We also can replace a by a term which depends on T^*, the period of production at the point $\bar{k}$ where $f(k)$ reaches its maximum. We then obtain

$$f(k)=\frac{1}{T^*(\psi-1)}\left\{\psi\bar{k}\left(\frac{k}{\bar{k}}\right)^{\frac{1}{\psi}}-k\right\}$$

Case 2: $0<\psi<1$. In that case $f''(k)<0$ implies that both, B and a are negative. We may then write $f(k)=\bar{a}k-\bar{B}k^{\frac{1}{\psi}}$ with $\bar{a}=-a$ and $\bar{B}=-B$

Here we no longer have a Cobb-Douglas function. Again, $f(k)$ reaches a maximum at a finite value of k.

We replace the constant of integration B by a term which is a function of $\bar{k}$, the value of k at which $f(k)$ reaches its maximum. The production function then reads:

$$f(k) = a\left\{k - \psi\bar{k}\left(\frac{k}{\bar{k}}\right)^{\frac{1}{\psi}}\right\}$$

We also can replace a by a term which depends on T^*, the period of production at the point $\bar{k}$ where $f(k)$ reaches its maximum. We then obtain

$$f(k) = \frac{1}{T^*(1-\psi)}\left\{k - \psi\bar{k}\left(\frac{k}{\bar{k}}\right)^{\frac{1}{\psi}}\right\}$$

For further reference below we observe that here we also can write

$$f(k) = \frac{1}{T^*(\psi-1)}\left\{\psi\bar{k}\left(\frac{k}{\bar{k}}\right)^{\frac{1}{\psi}} - k\right\}$$

which is exactly the same function as in case 1. Only here the two components of the product are negative. Yet their product is positive.

Case 3: $\psi = 1$. Here we have to integrate the differential equation in a different way. It is

$$\frac{dT}{dr} = \frac{\partial T}{\partial r} + \frac{\partial T}{\partial \theta} = (1-\psi)T^2 = 0$$

Thus the period of production T no longer depends on r. It is a constant across different steady states. For our exercise we then can treat it as a constant parameter. We remember that T is defined by

$$T = \frac{k}{w} = \frac{k}{f(k) - kf'(k)}$$

This yields the differential equation $f'(k)k = f(k) - \frac{k}{T}$

It can be integrated: $f(k) = \frac{C}{T}k - \frac{1}{T}k\ lnk + \frac{k}{T}$. Here C is a constant of integration. Indeed differentiating this equation with respect to k results in

$f'(k) = \frac{1}{T}\left\{C - lnk - \frac{k}{k} + 1\right\} = \frac{1}{T}\{C - lnk\}$

Therefore $f'(k)k = \frac{1}{T}\{Ck - k\ lnk\} = f(k) - \frac{k}{T}$ which is proof of a correct integration. We know that this production function reaches a maximum at a finite value of k, because the negative term rises faster with rising k than the positive terms. Let $\bar{k}$ be the value which maximizes $f(k)$. We then have $0 = f'(\bar{k}) = \frac{1}{T}\{C - ln\bar{k}\}$. Thus we can replace C by $ln\bar{k}$. Then our production function reads

$$f(k) = \frac{k}{T}\{ln\bar{k} - lnk + 1\} = \frac{k}{T}(1 + ln\frac{\bar{k}}{k})$$

Net output $f(k)$ per period is inversely proportional to the period of production. We can change the unit period, say from months to years, or, for that matter, to the length of the period of production T. Then, obviously the period of production takes on the value of unity. Instead, using the calendar year as our unit period, we can write

$$Tf(k) = k\{1 + ln\bar{k} - lnk\}$$

I now refer to one of the "Kaldor Facts", see Kaldor 1961 and, for example Jones 2016 . Empirically the capital output ratio has no trend. Apart from cyclical swings it basically is a constant through time. But the capital output ratio is almost the same as the ratio $\frac{k}{w}$. Remember that w is not only the wage rate, but all income except for the interest income on capital used in the production process. And the latter is a rather small proportion of national income. So the period of production $T = \frac{k}{w}$ basically behaves like the conventional capital output ratio. We then can rely on the empirical observation that T basically is a constant through time. On the other hand there is a clear downward trend in the risk-free real rate of interest. In our Solow model this means that $\frac{k}{\bar{k}}$ rises through time.

These two empirical observations help us to calibrate the value of ψ, the coefficient of intertemporal substitution.

We look at cases 1 ($\psi > 1$) and 2 ($0 < \psi < 1$) together. In both cases we have the production function

$$f(k) = \frac{1}{T^*(\psi - 1)}\left\{\psi\bar{k}\left(\frac{k}{\bar{k}}\right)^{\frac{1}{\psi}} - k\right\}$$

It then follows from $w = f(k) - kf'(k)$ that

$$T = \frac{k}{w} = T^*\left(\frac{k}{\bar{k}}\right)^{\frac{\psi-1}{\psi}}$$

Or, in logarithmic form,

$$lnT = lnT^* + \frac{\psi - 1}{\psi}\ln\left(\frac{k}{\bar{k}}\right)$$

Now, because of the near stationarity of T through time, in view of substantial shifts of $\left(\frac{k}{\bar{k}}\right)$ through time we can infer that the expression $\frac{\psi-1}{\psi}$ must be quite small in absolute value. This amounts to the conclusion that ψ must be close to unity.

We then arrive at

Theorem 8: Using the Solow model with a constant coefficient of intertemporal substitution ψ the empirical secular facts imply that ψ approximately equals unity.

As long as we are satisfied with the Solow approximation of the real world we then can use Case 3 with $\psi = 1$ as a good approximation of reality. The production function then is

$$Tf(k) = k\{1 + ln\bar{k} - lnk\}$$

Before we introduced the assumption of a constant , we derived the equation

$$\frac{dT}{dr} = \frac{\partial T}{\partial r} + \frac{\partial T}{\partial \theta} = (1 - \psi)T^2$$

If it were the case that ψ is substantially above unity everywhere then we do not need the assumption of a constant ψ to derive a contradiction with the secular fact of a constant capital output ratio. For then dT/dr would be consistently negative. And this in conjunction with the secular downward trend of r would have meant that T shows a secular upward trend. Thus, even without the assumption of a constant ψ we can be confident that ψ cannot be much higher than unity for most values of $k/\bar{k}$. In an analogous way we also can show that ψ cannot be much lower than unity. I believe that this result should be of relevance for the estimation of the traditional elasticity of substitution between labor and capital.

Section 8. Calibrating the Coefficient of Intertemporal Substitution for the Consumption System

A standard approach in modeling life-cycle consumption and saving patterns is the assumption that life time utility U of a person is an integral over a period utility function which itself depends on the per period consumption level in real (not nominal) terms. Thus one writes

$$U = \int_0^{a+b} u(\tilde{c}(t))dt$$

with $t = 0$ the beginning of adult life of the person; $a + b =$ expected number of years of adult life; $a =$ number of years in the labor force; $b =$ expected number of years in retirement; u the period utility; $\tilde{c}(t)$ = the level of consumption in terms of some appropriate commodity basket.

In this standard model I ignore time preference or other time (age) dependent influences on the period utility function. I also ignore planning for bequests. Time preference tends to reduce the propensity to save, whereas the wish to provide for one´s children after one´s death raises the propensity to save. The net effect then is likely to be substantially smaller than each of these two influences with opposite effects on saving.

Now I am interested in the time distribution of consumption for a given level of life utility U as a function of the rate of interest. This will then lead us to the coefficient of intertemporal substitution γ. For this purpose I simplify the analysis by assuming that the person´s supply of labor remains unaffected by a change in the rate of interest and the corresponding change in the wage rate. Obviously, this simplification implies that I underestimate the coefficient of intertemporal substitution.

I also assume that growth in this economy is only due to technical change. In other words: the population of the economy at large remains constant. It consists of overlapping generations, each of which has the same life utility function

We have a budget constraint: the present value of wages obtained by the consumer equals the present value of her/his consumption throughout adult life.

$$budget = \int_0^{a+b} \tilde{c}(t)e^{-rt}\, dt = \widehat{w}\ \frac{1-e^{(g-r)a}}{r-g}$$

Here g is the rate of growth of the annual wage and $\widehat{w}$ is the initial annual wage. A canonical formula for the period utility function uses the assumption of a constant relative risk aversion, denoted by μ. We then have

$$u\big(\tilde{c}(t)\big) = \frac{\tilde{c}^{1-\mu}}{1-\mu}, 0 < \mu; \mu \neq 1$$

or, for $\mu = 1$,

$$u(\tilde{c}(t)) = \ln(\tilde{c}(t))$$

Assuming maximization of life-time utility for a given budget, the intertemporal distribution of consumption then satisfies the condition that the marginal period utility of consumption is proportional to the present value of a monetary unit. Differentiation of u with respect to $\tilde{c}$ yields

$$u'(\tilde{c}) = \tilde{c}^{-\mu}$$

from which follows for the optimum consumption path

$$\tilde{c}(t)^{-\mu} = \tilde{c}(0)^{-\mu}e^{-rt} \text{ ,or, which is the same,}$$

$$\tilde{c}(t) = \tilde{c}(0)e^{\frac{rt}{\mu}}$$

Now we use the "naming convention" $\eta(r) = r$, which we used before. We then can write

$$\tilde{c}(t) = \tilde{c}(0)e^{\frac{\eta t}{\mu}}$$

and thus

$$e^{-rt}\tilde{c}(t) = e^{-rt}\tilde{c}(0)e^{\frac{\eta t}{\mu}}$$

We can get a better understanding of our approach, if we derive a formula for the partial derivative of Z with respect to η.

We have, by definition

$$Z = \frac{\int_0^{a+b} t\hat{c}(t)e^{-rt}dt}{\int_0^{a+b} \hat{c}(t)e^{-rt}dt} - \frac{\int_0^{a+b} t\hat{\lambda}(t)e^{-rt}dt}{\int_0^{a+b} \hat{\lambda}(t)e^{-rt}dt} = \frac{\hat{c}(0)\int_0^{a+b} te^{\frac{\eta t}{\mu}}e^{-rt}dt}{\hat{c}(0)\int_0^{a+b} e^{\frac{\eta t}{\mu}}e^{-rt}dt} - \frac{\int_0^{a+b} t\hat{\lambda}(t)e^{-rt}dt}{\int_0^{a+b} \hat{\lambda}(t)e^{-rt}dt}$$

or, which is the same,

$$Z = Z_c - Z_l$$

with Z_c the average time distance from zero of discounted period consumption and Z_l the average time distance from zero of discounted period labor supply

We differentiate this expression partially with respect to η at the point $\eta = r$

$$\frac{\partial Z}{\partial \eta} = \frac{1}{\mu}\frac{\left(\int_0^{a+b} t^2 e^{\frac{\eta t}{\mu}}e^{-rt}\,dt\right)\left(\int_0^{a+b} e^{\frac{\eta t}{\mu}}e^{-rt}\,dt\right)}{\left(\int_0^{a+b} e^{\frac{\eta t}{\mu}}e^{-rt}\,dt\right)^2} - \frac{1}{\mu}\frac{\left(\int_0^{a+b} te^{\frac{\eta t}{\mu}}e^{-rt}\,dt\right)^2}{\left(\int_0^{a+b} e^{\frac{\eta t}{\mu}}e^{-rt}\,dt\right)^2}$$

This expression can be written as

$$\frac{\partial Z}{\partial \eta} = \frac{1}{\mu}\, Variance\,\{\hat{c}(t)\}$$

By $Variance\,\{\hat{c}(t)\}$ I mean the second central moment of the distribution of the person´s consumption over the time axis. Its first moment is the time point of gravity of that consumption stream, which we earlier denoted by Z_c and which is the first term in the definition of Z.

By using these expressions for the flow of consumption in the budget equation we obtain for $\mu \neq 1$ and $r \neq g$

$$budget = \tilde{c}(0)\frac{1 - e^{(\frac{\eta}{\mu}-r)(a+b)}}{r - \frac{\eta}{\mu}} = \widehat{w}\frac{1 - e^{(g-r)a}}{r - g}$$

By L'Hopital's Rule we have for $r = g$ the equation

$$budget = \tilde{c}(0)\frac{1 - e^{(\frac{\eta}{\mu}-r)(a+b)}}{r - \frac{\eta}{\mu}} = \widehat{w}a$$

For the case of $\mu = 1$ (the logarithmic utility function) and $r = g$ we get, again by L'Hopital's Rule, upon partial differentiation with respect to η at $\eta = r$,

$$budget = \int_0^{a+b} e^{-rt}\tilde{c}(0)e^{\eta t}dt = \frac{1 - e^{(\eta - r)(a+b)}}{r - \eta}\tilde{c}(0) =$$

$$\frac{-(a+b)e^{(\eta - r)(a+b)}}{-1}\tilde{c}(0) = (a+b)\tilde{c}(0) = \widehat{w}a$$

We should note that the assumption of a constant relative risk aversion μ implies that, for any t the ratio $\tilde{c}(t)/\widehat{w}$ remains unaffected by a change in the initial wage rate $\widehat{w}$. Thus, also the waiting period Z does not depend on the value of $\widehat{w}$. But, of course, it depends on the value of the rate of interest.

Now I compute the coefficient of intertemporal substitution γ for this special household model. I specialize it further:

<u>First example:</u> Here I assume that μ is equal to unity, and $r = g$. We then have the logarithmic utility function and a rate of interest at the rate of growth

As derived above we then have the equation

$$\tilde{c}(0)(a+b) = \widehat{w}a$$

This implies a waiting period $Z = \frac{b}{2}$.

Now we note that, after optimization by the consumer, any feasible marginal change $d\hat{c}(t)$ in the period consumption path must have the property

$$\int_0^{a+b} e^{-rt}d\tilde{c}(t)\,dt = 0$$

In particular then

$$\int_0^{a+b} e^{-rt}\frac{\partial\tilde{c}(t)}{\partial\eta}dt = 0$$

In the special case of the logarithmic utility function we obtain from

$$e^{-rt}\tilde{c}(t) = e^{-rt}\tilde{c}(0)e^{\eta t}$$

that at $\eta = r$

$$\frac{\partial \tilde{c}(0)}{\partial \eta} = -\tilde{c}(0)\frac{\int_0^{a+b} tdt}{\int_0^{a+b} 1dt} = -\tilde{c}(0)\frac{(a+b)^2}{2(a+b)} = -\tilde{c}(0)\frac{a+b}{2} = -\tilde{c}(0)Z_c$$

From this follows for $\eta = r = g$ and $\mu = 1$

$$\frac{\partial Z_c}{\partial \eta} = Variance\{e^{-rt}\tilde{c}(0)e^{\eta t}\} = \frac{(a+b)^2}{12}$$

We then can compute γ, the coefficient of intertemporal substitution:

$$\gamma = -\frac{\partial \frac{1}{Z}}{\partial \eta} = -\frac{d\frac{1}{Z}}{dZ}\frac{\partial Z_c}{\partial \eta} = \frac{1}{Z^2}\frac{(a+b)^2}{12} = \frac{(a+b)^2}{12\frac{b^2}{4}} = \frac{4}{12}\left(\frac{a+b}{b}\right)^2$$

As a reasonably realistic numerical example we assume that b is half as large as a. We then get the result

$$\gamma = \frac{4}{12}9 = 3$$

<u>Second example:</u> We assume relative risk aversion to be $\mu = 2$. In the empirical literature different point estimates of μ have been found. It appears that μ declines with rising income: risk aversion becomes weaker as wealth rises. But there seems to be agreement that μ is at least as high as unity. Further we assume $r = g = 0$. Above we already have derived

$$budget = \tilde{c}(0)\frac{1 - e^{(\frac{\eta}{\mu} - r)(a+b)}}{r - \frac{\eta}{\mu}} = \widehat{w}a$$

which, for $r = 0$ and $\eta \neq 0$ reads

$$budget = \tilde{c}(0)\frac{1 - e^{\frac{\eta}{\mu}(a+b)}}{-\frac{\eta}{\mu}} = \widehat{w}a$$

By L´´Hôpital´s Rule this means for $\eta = r = 0$

$$budget = \tilde{c}(0)\frac{-(a+b)\frac{1}{\mu}}{-\frac{1}{\mu}} = \tilde{c}(0)(a+b) = \widehat{w}a$$

We then have $Z = \frac{1}{b}$

Partial differentiation with respect to η generates

$$\frac{\partial Z}{\partial \eta} = \frac{1}{\mu}\, Variance\, \{\tilde{c}(t)\} = \frac{1}{\mu}\frac{(a+b)^2}{12}$$

For my calibration exercise I assume μ=2. I again assume that a, the number of years spent in the labor force is twice the number of pensioners´ years b. We then obtain

$$\gamma = -\frac{\partial \frac{1}{Z}}{\partial \eta} = -\frac{d\frac{1}{Z}}{dZ}\frac{\partial Z_c}{\partial \eta} = \frac{1}{Z^2}\frac{1}{2}\frac{(a+b)^2}{12} =$$

$$\frac{1}{2}\frac{(a+b)^2}{12\frac{b^2}{4}} = \frac{4}{24}\left(\frac{a+b}{b}\right)^2 = \frac{1}{6}9 = 1.5$$

<u>Conclusion.</u> Empirically the average value of μ is between 1 and 2. From our two examples we may surmise that the coefficient of intertemporal substitution, γ, is between 1.5 and 3. But in our second example we assume an unrealistically low rate of growth g of zero. And for $r = g > 0$ a risk aversion parameter of $\mu = 2$ would yield a value of γ below 1.5. This reduction may be compensated by a realistically somewhat smaller risk aversion parameter $\mu < 2$. Moreover, we have ignored that with life utility remaining the same, at a higher interest rate labor supply may change in the direction, which raises the coefficient of intertemporal substitution.

So I believe that we do not overestimate γ when we put it equal to 1.5.

We conclude that the CIS in the consumption system is likely to be higher than the CIS in the production system. This result also can be made plausible by reference to the high life expectation in modern times. As we noted earlier in Chapter 2, Section 5, we have the equation

$$\frac{dZ}{dr} = Variance\, \{e^{-rt}\lambda(t)\} - Variance\, \{e^{-rt}\gamma(t)\}$$

If the proportion of the two "Variances" remains the same this formula indicates that the "leverage" of the rate of interest on the time structure of consumption rises in proportion to the square of the standard deviation of consumption across the time axis. And the latter can be estimated to rise in proportion to the life expectation. Actually, the pension period is the main driver of the difference between the two variance terms. And it, so far, has risen more than in proportion to life expectation. So the high life expectation in the OECD world and beyond is responsible for the strong leverage of the rate of interest on the time structure of consumption.

<u>Section 9. Estimating the total proportionate loss of welfare due to a steady state rate of interest different from the rate of growth</u>

For our meta-model we have a formula for the second degree Taylor approximation of welfare loss due to $r \neq g$. According to Theorem 7 the proportionate welfare loss is given by

$$\Omega = \frac{1}{2}(\psi T^2 + \gamma Z^2)(r-g)^2$$

Now, we have an estimate from section 7 of $\psi = 1$ and from section 8 of $\gamma = 1.5$. Earlier, in Chapter 2, section 6 we have used the following values for the period of production T, at $r = g$; and for the waiting period Z at $r = g$. There we assumed $T = 5\ years$ and $Z = 10\ years$. We assumed that the natural rate of interest, ρ, was five percentage points below the rate of growth.

Using these values the proportionate loss in welfare can be assessed:

$$\Omega = \frac{1}{2}(1 times\ 25 + 1.5\ times\ 100)(5\%)^2 = \frac{1}{2}175 times\ 0.0025 = 21.875\%$$

Using these numbers the steady state without public debt has a standard of life which is roughly 22 % below the standard of life at a public debt period $D = Z - T = 5\ years$. For, using Theorem 3, at that public debt period, we have a rate of interest $r = g$, which corresponds to the Golden Rule steady state.

It so happens that this value for Ω almost exactly equals the proportionate welfare loss, which we obtained in Chapter 2, Section 6, by the assumption that the envelope period of production and the envelope waiting period are constant across different steady state rates of interest.

Obviously, the welfare loss there is linear in terms of the difference $g - \varrho$, whereas the second degree Taylor approximation Ω is quadratic in terms of the difference $g - \rho$. So it must be seen as a coincidence that the "realistic" values for $g, \rho, \psi, \gamma, T\ and\ Z$ lead to the same result in the linear and in the quadratic case.

What is perhaps of some interest is the fact that the welfare loss derived from the production system is substantially smaller than the welfare loss derived from the consumption system. The latter is at least four times larger than the former, using our numerical example. The Phelps-Weizsäcker Golden Rule is less important than the Samuelson “Golden Rule”. Note that the opposite would be true, if the natural rate of interest were above the rate of growth. For then Z would be smaller than T at $r = g$.

Using any specific model of the economy, one can of course assess, how good the Taylor approximation of the welfare loss is, if compared with the “true” model. Here we do not go into the details, as this would require a lot of effort, time and space. Just, as an appetizer, I provide two results for the comparison of the “true” model with the corresponding Taylor approximation. First I use the model for the production sector described in Section 7. Next I use the logarithmic utility function, discussed in Section 8.

The following table gives the results for the production model of Section 7. Given the period of production $T = 5\ years$, we provide the welfare loss, using the model of Section 7 for $g - r = 1\%, 2\%, 3\%, 4\%$ and 5%. And we compare this with the Taylor approximation, using the same coefficient of intertemporal substitution $\psi = 1$, as in the “true” model.

Table 1 Welfare loss in the production sector due to $r \neq g$

Table	Welfare Loss "True Model"	Welfare Loss Taylor apr. (TA)	Error of TA % of true Loss
g-r= 1%	0,1292%	0,1250%	-3,29%
g-r= 2%	0,5346%	0,5000%	-6,48%
g-r= 3%	1,2441%	1,1250%	-9,57%
g-r= 4%	2,2878%	2,0000%	-12,58%
g-r= 5%	3,6981%	3,1250%	-15,50%

As is well known, a Taylor approximation of any finite degree is more accurate for small deviations than for larger ones. Up to an interest rate deviation of 25 % per period of production, $(g - r)T$, the error is tolerable. I believe, even after decades of empirical macro-economic research, economists are not so sure about the “true welfare” loss due to a risk-free rate of interest that deviates from the steady state rate of growth.

In this particular case the “true” welfare loss turns out to be larger than its second degree Taylor approximation, derived from the “meta-model”. At this time I do not know how general this result is, but it reappears in the second example: the logarithmic utility function.

For this example I use the logarithmic utility function $(\mu = 1)$ and the same assumptions as in Section 8: the waiting period Z at $r = g$ is $10\ years$. And the ratio between the life span with earned wages, a, and the pension life span, b, is equal to 2, so that $\frac{b+a}{b} = 3$. The result is contained in the following table

<u>Table 2 welfare loss in the consumption sector due to $r \neq g$</u>

Table	Welfare loss "True Model"	Welfare loss Taylor appr (TA)	Error of TA % of true loss
g-r= 1%	1,5068%	1,5000%	-0,4489%
g-r= 2%	6,1089%	6,0000%	-1,7831%
g-r= 3%	14,0574%	13,5000%	-3,9653%
g-r= 4%	25,7884%	24,0000%	-6,9351%
g-r= 5%	41,9520%	37,5000%	-10,6121%

In this case the TA error is somewhat smaller percentagewise than in the example from the production system. But the welfare loss of the “true model” is substantially larger than in the example from the production system. A deviation of the annual interest rate from the annual growth rate of five percentage points leads to a loss in welfare of more than 40 %. This, I believe, is mainly because the waiting period Z is twice the period of production. In terms of the rate of interest for the pertinent periods T or Z the deviation in the production system amounts to $T(g - r)$, which is 25 %, because the period of production is five years. In terms of the rate of interest for the waiting period the deviation amounts to $Z(g - r)$ which is 50 %, because the waiting period is ten years. In the Taylor approximation the pertinent periods T and Z enter with their square, so that 50 % deviation has the fourfold effect of the 25 % deviation. Given that the second degree Taylor approximation is not so bad, this is indicative for the observation that the Samuelson Golden Rule is so much more important than the Phelps-Weizsäcker Golden Rule.

Chapter 4 Capitalized Rents

Future Ricardian land rents can be capitalized. At any given moment of time they are then a further wealth item of personal wealth beyond the produced capital goods tied down in the production process ($v(r;\theta)$) and public debt (wD). Other rents can also be capitalized: monopoly rents, reputation rents in the form of customer loyalty etc.

Let $q(r;\theta)$ be the rent income per worker, so that we can write

$$y = x(r;\theta) + rv(r;\theta) + q(r;\theta)$$

Here $x(r;\theta)$ is the "rest of income", after deduction of the risk free return on capital and of rent income. It contains wages, includingthe return on human capital, which we do not subsume under rent income or $v(r;\theta)$, because human capital is not part of the capital market. The variable $x(r;\theta)$ also contains what economics calls "imputed wages" of self-employed persons. The variable $x(r;\theta)$ contains all income, which, by its intrinsic nature, or by custom or by law, cannot be capitalized on the capital market. We maintain the definition of $w(r;\theta) = y - rv(r;\theta) = x(r;\theta) + q(r;\theta)$

We now consider the capitalized value of future rent income. Let $\omega = \frac{q}{w}$ be the share of rent income in $w(r;\theta)$. Let $Lw(r;\theta)$ be the market value of capitalized rents. The variable L obviously has the dimension "time". We decompose L into two components: $L = \omega l(r;\theta)$. Since ω is dimension-free the variable $l(r;\theta)$ again has the dimension "time".

Private wealth $\hat{v}$ per worker then is

$$\hat{v} = v(r;\theta) + w(r;\theta)(\omega l(r;\theta) + D)$$

The variable $l(r;\theta)$ is a kind of "reliability" indicator of future rent income. It is a macro-economic average of quite distinct reliability indicators. Rent income from patent licenses ends with the expiration of the patent. Rent income from some piece of land may change upwards or downwards, depending on the particular circumstances surrounding this piece of land. Rent income from a particular urban piece of land, used for dwellings rented out to dwellers, may have a high reliability at large, but changes in the law concerning rent control may shift the land rent from the owner to the dwellers. The risk of law changes at the expense of landowners has a negative impact on the value of the "reliability indicator".

The Golden Rule of Accumulation remains valid even with the presence of rent income. As before, we can write down the equation

$$c(\theta) + gv(r;\theta) = w(r;\theta) + rv(r;\theta)$$

Partial differentiation with respect to θ yields

$$\frac{dc}{d\theta} + g\frac{\partial v}{\partial\theta} = \frac{\partial w}{\partial\theta} + r\frac{\partial v}{\partial\theta}$$

Let θ^* be the value of θ, which maximizes consumption within the set $Theta$. Thus, at θ^* we have $\frac{dc}{d\theta} = 0$. Assuming, as in Chapter 2, that r is an unbiased price signal (Assumption 4 in Chapter 2) we obtain at $\theta(r) = r$, the equation $\frac{\partial w}{\partial\theta} = 0$. Hence, at θ^* we have the equation

$$0 + g\frac{\partial v}{\partial\theta} = 0 + r\frac{\partial v}{\partial\theta} = 0 + \theta^*\frac{\partial v}{\partial\theta}$$

By Assumption 5 in Chapter 2 („law of demand) we have the inequality $\frac{\partial v}{\partial\theta} < 0$. It follows

$$\theta^* = g$$

There exists the theoretical possibility that capitalized rents prevent that Assumption 1 is fulfilled: that the set $Theta$ does not contain the value of the rate of growth g, because for any $r \in Theta$ the inequality $r > g$ holds; see for example Homburg 1991. Here, in this paper we do not discuss this case further, because we consider it unrealistic.

As above, in Chapters 2 and 3, we can define the "waiting period" by means of the equation

$$\frac{\partial U}{\partial r} = \frac{\partial U}{\partial\widehat{w}}Z\widehat{w}$$

Also, Theorem 2 remains unaffected by the introduction of capitalized rents, because steady state consumption compatible with a given level of life utility U obtains its minimum at the rate of interest $r = g$ (Samuelson Golden Rule).

Now, we look at private wealth for the case $r = g$.

$$\hat{v}\big(g;\eta(g)\big) = v(g;\theta(g)) + w(g;\theta(g))[D + \omega l(g;\theta(g)]$$

But we know from Chapter 2 that $v\big(g;\theta(g)\big) = w\big(g;\theta(g)\big)T$. Thus, we get

$$\hat{v}\big(g;\eta(g)\big) = w(g;\theta)[T + D + \omega l]$$

On the other hand, because of Theorem 2 and because of $\widehat{w}\big(g;\eta(g)\big) = w(g;\theta(g))$ we have

$$0=\frac{dU}{d\widehat{w}} = \frac{\partial U}{\partial \widehat{w}}\frac{\partial \widehat{w}}{\partial r} + \frac{\partial U}{\partial r} = \frac{\partial U}{\partial \widehat{w}} w[-T - D - \omega l + Z]$$

This means

$$T + \omega l \ = Z - D;$$

We may call this equation the extended fundamental equation of steady state capital theory. Moreover we can write

$$\hat{v}\big(g;\eta(g)\big) = w\big(g;\theta(g)\big)Z = c(g)Z$$

For given values of the waiting period Z and the period of production T the public debt period D and ωl are “perfect substitutes”. Government policy can influence both variables. As an example: compared with its absence a forceful anti-trust policy may reduce rent share ω in “other income”. If government wants to maintain $r = g$ it may then want to raise the government debt period D.

Chapter 5. Two Case Studies on Technical Progress

Section 1 Embodied Technical Progress

In Solow et al 1966 we develop a model of embodied technical progress, i.e. a model of “vintage capital”. Labor and machines produce output. Machines have a “vintage” which is the moment of time when they have been made. A machine of vintage τ requires $e^{-g\tau}$ labor years to produce one unit of output per year. The exogenously given parameter g is the rate of technical progress. I choose the time unit “year” so as make the “annual” output of a machine equal to unity, i.e. equal to the building cost of the machine. If the output of the machine per calendar year is equal to q then our artificial “year” equals $1/q$ calendar years. This choice of the unit period has the advantage that we only have to deal with a single exogenous parameter, namely the rate of technical progress g. For ease of presentation I assume that the available labor force of the economy, L, is constant through time. But, the results also generalize to a labor force that grows or shrinks exponentially.

In my present exercise, I only look at steady states. In the original 1966 paper we also investigated deviations from the steady state. We showed that the equilibrium real rate of interest $r(t)$ was an unbiased price signal of the social rate of intertemporal substitution of consumption at different moments of time. Therefore, the model *a fortiori* has the property of an unbiased price signal on the production side (UBPS1) which we introduced in Chapter 2 of this paper. It is straightforward to show that our 1966 vintage model also satisfies Assumptions 1, 2, and 3 of the meta-model described in Chapter 2 above.

In my present exercise I do not investigate the consumption/saving behavior of households. For my purposes it is then allright simply to assume that the coefficient of intertemporal substitution on the consumption side is equal to zero. Thus $\gamma = 0$. We then can apply Theorem 1 above to show that the Golden Rule applies for the comparison of different steady states.

Moreover, applying the meta-model of the present paper, the vintage model of this section implies the following formula for the second order Taylor approximation. Let c be steady state consumption per unit of labor. Then the second order Taylor approximation is

$$c(r) \approx c(g)(1 - \psi T^2 (r-g)^2)$$

Here, as in the meta-model, T is the period of production, evaluated at the Golden Rule Steady state.

On the other hand, we can find the second order Taylor approximation of consumption per head by direct computation in the Solow et al 1966 paper. The comparison of both ways to arrive at a result, which must be identical, enables us to compute the value of the coefficient of intertemporal substitution ψ in this specific model.

The economy in this model has an exogenously given steady state rate of growth equal to $g > 0$, the rate of technical progress. At time t a given machine of vintage $\tau \leq t$ has an output of unity and labor costs of $w(t)e^{-g\tau}$. In a steady state the wage rate grows at the rate of technical progress. So we can write

$$w(t) = w(0)e^{gt}$$

Thus, the wage bill of the vintage τ machine at time t equals

$$wage\ bill = w(t)e^{-g\tau} = w(0)e^{g(t-\tau)}$$

In the long run the wage bill for a machine of vintage τ would be higher than unity – and it would run with an operating loss. At the time the wage bill reaches unity the machine will be scrapped. We then can compute the "quasi-rent" of a machine of vintage $\tau = 0$. Let a be the age of machine of vintage $\tau = 0$ at which its wage bill reaches unity. Thus, a is defined by the equation

$$1 = w(0)e^{ga} \text{ or, which is the same: } 0 = \ln\big(w(0)\big) + ga$$

Let $q(t;\tau)$ be the quasirent a machine of vintage τ receives at time t. We then have

$$q(t;\tau) = 1 - w(0)e^{g(t-\tau)} \text{ for } \tau \leq t \leq \tau + a$$

$$q(t;\tau) = 0 \text{ otherwise}$$

In market equilibrium, the sum of the present values of the quasirents at time τ must be equal to the investment cost of the machine, i.e. it must be equal to unity. We thus have

$$1 = \int_{\tau}^{\tau+a} e^{-r(t-\tau)}\, q(t;\tau)dt = \int_0^a e^{-rt}\, q(t;0)dt$$

Hence for $r \neq 0$ and $r \neq g$

$$1 = \frac{1-e^{-ra}}{r} - w(0)\frac{1-e^{-(r-g)a}}{r-g}$$

For $r = g$ L´Hopital´s rule yields

$$1 = \frac{1-e^{-ga}}{g} - w(0)a = \frac{1-e^{-ga}}{g} - ae^{-ga} = \frac{1}{g} - e^{-ga}\left\{\frac{1}{g} + a\right\}$$

I now compute the Golden Rule value of the rate of interest r. Consumption per head, c, is the difference between gross output per head and gross investment per head. Thus, we have

$$c = \frac{1}{a}\int_{-a}^{0} e^{g\tau}\, d\tau - \frac{1}{a} = \frac{1}{a}\left\{\frac{1-e^{-ga}}{g} - 1\right\}$$

Putting the first derivative of c with respect to a equal to zero implies

$$\frac{dc}{da} = \frac{-1}{a^2}\left\{\frac{1-e^{-ga}}{g} - 1\right\} + \frac{1}{a}e^{-ga} = 0$$

which implies $1 = \frac{1}{g} - e^{-ga}\left\{\frac{1}{g} + a\right\}$ which is the equilibrium condition for a rate of interest equal to the rate of growth. That´s the Golden Rule.

Now we can compute the second order Taylor approximation for consumption per head around the maximum point, as a function of the rate of interest.

Consumption per head is a function of a, the life span of the machines. But a is a function of the steady state rate of interest.

We thus can compute the first derivative of c with respect to the rate of interest r.

We obtain $\frac{dc}{dr} = \frac{dc}{da}\frac{da}{dr}$. The second derivative then is

$$\frac{d^2c}{dr^2} = \frac{d^2c}{da^2}\frac{da}{dr} + \frac{dc}{da}\frac{d^2a}{dr^2}$$

For the computation of the second order Taylor approximation, we only need the second derivative at the point $r = g$. And here $\frac{dc}{dr} = 0$ and therefore $\frac{dc}{da} = 0$, because we always have $\frac{da}{dr} > 0$. We then only have to compute $\frac{d^2c}{da^2}\frac{da}{dr}$.

We have

$$\frac{d^2c}{da^2} = \frac{ga^2\{-ge^{-ga}(1+ga) + ge^{-ga}\} - 2ga\{numerator\ of\ \ dc/da\}}{g^2a^4}$$

But at $r = g$ the numerator of the first derivative is zero, so that here it boils down to

$$\frac{d^2c}{da^2} = \frac{-ge^{-ga}(1+ga) + ge^{-ga}}{ga^2} = -\frac{g}{a}e^{-ga} < 0$$

We now compute $\frac{da}{dr}$. We know that $\frac{dw}{da} = -ge^{-ga} = -gw$. On the other hand we also know from the meta-model that $\frac{dw}{dr} = -Tw$ where T is the period of production in our economy

So we obtain

$$\frac{da}{dr} = \frac{da}{dw}\frac{dw}{dr} = \left(-\frac{1}{gw}\right)(-Tw) = \frac{T}{g}$$

This leads to the formula at $r = g$

$$\frac{d^2c}{dr^2} = \frac{d^2c}{da^2}\frac{da}{dr} = \frac{-gw}{a}\frac{T}{g} = -\frac{wT}{a}$$

Taking account of the equation $w = c(g)$ we then obtain for the second order Taylor approximation around $r = g$

$$c(r) \approx c(g) + \frac{1}{2}\frac{d^2c}{dr^2}(r-g)^2 = c(g) - \frac{1}{2}\frac{wT}{a}(r-g)^2$$
$$= c(g)\left\{1 - \frac{1}{2}\frac{T}{a}(r-g)^2\right\}$$

Comparing this formula with the corresponding formula of the meta-model

$$c(r) \approx c(g)(1 - \frac{1}{2}\psi T^2(r-g)^2)$$

implies

$$\frac{T}{a} = \psi T^2 \text{ or } \psi = \frac{1}{aT}$$

We should note that ψ is dimension free. How does this square with the present formula? It does, because in our special set-up we chose the unit period so that per period gross output per machine is equal to unity. Thus both, a and T are ratios between two time periods and thus are dimension free. If, instead we had chosen the calendar year as our unit period the numerator in the expression for ψ would not be unity but rather the square of the time period needed for a machine to produce gross output equal to its own cost of production.

A numerical evaluation of ψ for realistic values of the rate of technical progress implies that it is between 0.3 and 0.4.

The 1966 vintage model did not contain substitution between labor input and machines of a given vintage. Obviously, if one would add such substitution the resulting coefficient of intertemporal substitution would be larger than in the model discussed here. For a vintage model with such substitution see Sheshinski 1969.

From the equation

$$\frac{dc}{da} = \frac{-1}{a^2}\left\{\frac{1-e^{-ga}}{g} - 1\right\} + \frac{1}{a}e^{-ga} = -\frac{1}{ga^2}\{1 - g - e^{-ga}(1+ag)\}$$

we see that $\{1 - g - e^{-ga}(1 + ag)\}$ is a monotonically decreasing function of a rising value of a. We know already that this expression is zero at $r = g$. Thus, together with $\frac{da}{dr} > 0$ we get the result that

$$\frac{dc}{dr}(g - r) = \frac{dc}{d\theta}(g - \theta) \geq 0$$

Therefore, we can apply the "Almost Converse of Theorem 1" of our meta-model to conclude for the 1966 vintage model

$$\frac{1}{c(\theta)}\frac{dc}{d\theta} \geq \frac{1}{v(r;\theta)}\frac{\partial v(r;\theta)}{\partial \theta},$$

which is our "law of demand" for a steady state economy (Assumption 5).

In Chapter 2, Section 8, above I show the basic duality relation of steady state capital theory: for any given technique θ the wage-interest curve is the same as the consumption-growth tradeoff curve applicable with the same technique. From this follows, for example, the insight that stimulating growth, beneficial as it may be, is not necessarily a method to get out of the conundrum of the negative equilibrium rate of interest. If the consumption-growth tradeoff curve is convex (which corresponds to the "neoclassical" Wicksell effect) then, for a given rate of interest (and hence a given technique) a higher growth rate means that the economy needs less capital.

We can apply this insight to the vintage model discussed in this section. It is easy to show that the consumption-growth tradeoff curve for a given technique is convex in this model. Here a given technique means a given life span a of the machines. So I return to the original version of the steady state consumption

$$c = \frac{1}{a}\int_{-a}^{0} e^{g\tau}\, d\tau - \frac{1}{a}$$

Keeping a constant I differentiate with respect to g. We have

$$\frac{dc}{dg} = \frac{1}{a}\int_{-a}^{0} \tau e^{g\tau}\, d\tau < 0$$

Differentiating again yields

$$\frac{d^2c}{dg^2} = \frac{1}{a}\int_{-a}^{0} \tau^2 e^{g\tau} d\tau > 0$$

Thus, we have shown that the consumption-growth tradeoff curve for a given technique is convex. This means that for a constant rate of interest a policy, which stimulates growth, reduces the demand for capital in the production sector. Note that the result does not depend on our assumption of fixed coefficients: Even if substitution between labor and capital for machines with a given vintage were possible, the same result would obtain, because, keeping the rate of interest fixed means that there is no substitution.

Due to the duality relation of steady state capital theory it then appears that vintage models generally have the property of a "neoclassical Wicksell effect".

Section 2 A Model of Induced Technical Progress

In this section I present a simple macro-model of "learning by doing", cf. Arrow (1962). Consider an economy in which we stipulate existence of a variable Q, which we call "cumulative knowledge". It obeys the following growth law

$$\frac{dQ}{dt} = \alpha C, \alpha > 0$$

Here C is the value of total consumption in this economy. As in the pioneering Arrow paper of 1962, it is induced technical progress.

As in the case of Harrod-neutral technical progress we have a concept of labor inputs in the form of "labor-efficiency units". Let A be the availability of labor in efficiency units: It is the product of physical labor L and their productivity factor, which we assume to be $e^{\beta t}Q^{\mu}$ with $\beta > 0$ and $0 \leq \mu < 1$. In a formula

$$A = Le^{\beta t}Q^{\mu}$$

Output Y in this economy then is

$$Y = Af(k) = Le^{\beta t}Q^{\mu}f(k)$$

Here $f(k)$ is a Solow production function per efficiency unit of labor; and k is "capital intensity" per labor efficiency unit. The total capital stock then is

$$K = Ak = Le^{\beta t}Q^{\mu}k$$

Now we can show that there are different steady states with different marginal productivities of capital $f'(k)$, but with the same rate of growth g. Provided that L remains constant through time, this rate of growth is this

$$g = \frac{\beta}{1-\mu}$$

The level of cumulative knowledge in a steady state is given by

$$Q = \alpha \frac{1-\mu}{\beta} C = \frac{\alpha}{g} C$$

In addition, consumption C is of course $C = Y - gK$

Indeed, if we use these equations we find: k is constant through time; and thus marginal productivity of capital $\frac{df}{dk}$ is constant through time. All macro-economic variables, except L grow at the same rate $g = \frac{\beta}{1-\mu}$.

We have

$$\frac{dA}{dt}\frac{1}{A} = \beta + \mu \frac{dQ}{dt}\frac{1}{Q} = \beta + \mu \frac{\beta}{1-\mu} = \frac{\beta}{1-\mu}$$

$$\frac{dY}{dt}\frac{1}{Y} = \frac{dA}{dt}\frac{1}{A} = \frac{\beta}{1-\mu}$$

$$\frac{dK}{dt}\frac{1}{k} = \frac{dA}{dt}\frac{1}{A} = \frac{\beta}{1-\mu}$$

$$\frac{dC}{dt}\frac{1}{C} = \frac{dY}{dt}\frac{1}{Y} = \frac{\beta}{1-\mu}$$

$$\frac{dQ}{dt}\frac{1}{Q} = \frac{\alpha C}{Q} = \frac{\alpha C}{\alpha \frac{1-\mu}{\beta} C} = \frac{\beta}{1-\mu}$$

It is a consistent system of equations. The steady state rate of growth is connected with the rate of exogenous technical progress β by a "multiplier" $\frac{1}{1-\mu}$. Its value rises with the parameter μ, which stands for the power of endogenous technical progress, i.e. with the power of learning by doing. Like in Arrows 1962 paper, we observe macro-economic economies of scale. This can be seen by differentiating Y with respect to L

$$\frac{\partial Y}{\partial L}\frac{L}{Y} = 1$$

But Y also rises with rising K and rising Q, so that for

$\frac{dL}{L} = \frac{dK}{K} = \frac{dQ}{Q}$>0 and $f'(k) \geq 0$

we get

$$\frac{dY}{Y} = \frac{dL}{L} + \frac{f'(k)}{k}\frac{1}{A}\frac{dK}{K} + \mu\frac{dQ}{Q} > \frac{dL}{L} = \frac{dK}{K} = \frac{dQ}{Q}$$

The percentage addition of output Y is greater than the proportional percentage addition of its inputs. This is a case of economies of scale. It also implies that the input remunerations cannot all equal their marginal social product.

If we stick to Assumption 4 of our meta-model (UBPS1), we have $r = f'(k)$. This means that the other inputs, in particular labor, receive less than their marginal social product. Here we then may have inefficiencies, which pervade all the steady states. Below in Chapter 7 we deal with situations where Assumption 4 does not hold.

Chapter 6. Intertemporal Change of the Waiting Period and the Period of Production

Section 1. The Generalized Theorem 2 – That Life Utility is Maximized at the Rate of Interest Equal to the Rate of Growth

Our steady state analysis in Chapters 2, 3, 4 and 5 rested on the Assumptions 2 and 7 that, for a given rate of interest, the steady state exhibited a stationary waiting period Z and a stationary period of production T. In this Chapter, we analyze secular trends in these two periods. The most evident (and most important) case is the secular rise in the waiting period. It is due to the secular rise in material wealth and in life expectancy. Here, in this paper, we do not investigate the empirical side of these secular trends. We simply take them as given.

Let us look at the provision for the time as pensioner. In the simple example of a work life of a years and an expected pensioner life of b years and a constant stream of consumption over these $a + b$ years and a constant stream of wage income over the work life, a zero interest rate yields a waiting period $Z = \frac{b}{2}$. If life expectancy rises and expected pensioner life time b rises from one age cohort to the next one we expect a secular upward trend of Z. Even if with rising life expectancy the working period and the pensioner period grow in tandem it would still be the case that there is a secular upward trend of the waiting period $Z = \frac{b}{2}$.

In the production system, it is not so clear what we have to expect for the capital-output ratio at a given rate of interest. A feeling is widespread that digitization and, in particular, artificial intelligence will eliminate many jobs without a sufficient number of replacement jobs. For the economist this sounds like a labor-saving bias in future technical progress. On the other hand, there are predictions that digitization and, in particular, artificial intelligence are likely to lead to a more intensive use of capital equipment: ease of car sharing as an example, or Airbnb. For the economist this sounds like capital-saving technical progress. At this time, we may be unable to predict the future bias in technical progress. Nevertheless, it is useful to find out, which policy would be an appropriate answer to trend biases in technical progress.

I look at a rate of interest equal to the rate of growth: $r = g$. I assume that the "benevolent dictator" wants to maintain that rate of interest. We know that then in equilibrium $\hat{v} = \hat{w}Z = \hat{w}(T + D)$. Now I assume that Z grows with time. Let

$$\frac{dZ(g; \eta(\overline{U}(g); g)}{dt} = \varepsilon_2$$

For the moment I assume that $T(g; \theta(g))$ remains constant through time. To maintain equilibrium at $r = g$, hence $Z = T + D$, we therefore need to grow the public debt period by

$$\frac{dD(g)}{dt} = \varepsilon_2$$

For a given primary surplus $(r - g)D$ government debt rises then by $\hat{w}gD + w\varepsilon_2$ which implies $\frac{dD}{dt} = \varepsilon_2$, because $\hat{w}$ rises at the rate g.Thus, the "tax" on consumers is $\hat{w}((r - g)D - w\varepsilon_2 = -w\varepsilon_2$. This then implies for $r = g$ the equation

$$\hat{w} = w + w\varepsilon_2 = w(1 + \varepsilon_2)$$

On the other hand, the consumption sector wants to save beyond maintaining the "waiting period". Indeed these additional savings amount to wε_2. Thus, the consumption level of the consumption sector remains unaffected by the desire to raise $\hat{v}$ - provided the government accommodates this desire by incurring a corresponding rise in its government debt period, so that the equilibrium rate of interest remains stable through time.

Similarly, we carry through the government debt answer to changes in the period of production. Assume that with a constant rate of interest $r = g$ the period of production changes through time, so that

$$\frac{dT(g;\theta(g))}{dt} = \varepsilon_1$$

In order to maintain the equilibrium condition $Z = T + D$ the rate of change of the public debt period needs to be

$$\frac{dD}{dt} = -\varepsilon_1 + \varepsilon_2$$

Our welfare analysis of Chapter 2, Section 4 (Theorem 2) remains valid. This then is what we call

<u>Theorem 2- generalized: In a comparison between different steady states, each with a rate of interest remaining constant through time, life-time utility U is highest with a rate of interest equal to the rate of growth of the economic system, even if structural parameters T and Z are not necessarily constant through time.</u>

<u>Sketch of Proof:</u> The basic idea is that in case of a government debt policy, which stabilizes the rate of interest the additional (positive or negative) net debt growth beyond maintaining the public debt period exactly corresponds to the additional (positive or negative) desire of consumers to save beyond maintaining the actual ratio between private wealth to "wage" income. Therefore, actual consumption by private persons is the same as in the case of $\hat{w}/\hat{v}$ remaining constant through time. A rising "waiting period" Z is equivalent to a rising level of wealth $\hat{v}$, accomplished by a corresponding higher savings rate, which is "financed" by the lower tax on consumers, because the government raises its debt level above the one that stabilizes the debt period D – so as to thereby maintain the equilibrium rate of interest r=g.

And a rising (or falling) period of production is equivalent to a rising (or falling) amount of capital per worker, v. Consumers want to compensate this for given Z and given $\hat{v}$ by a correspondingly lower (or higher) amount of government debt bonds in their wealth portfolio. The government provides for this change in demand for government debt by its accommodating supply of government debt bonds. Thereby demand and supply of government debt remain in equilibrium at the stable rate of interest $r = g$. And consumption is not affected by the changes in Z and/or T.

Using the same basic idea for any steady state rate of interest $r \in R$ we also can show that, by means of an accommodating government debt policy, changes through time of the corresponding levels of $\hat{v}$ (and corresponding waiting period Z) and of the corresponding level of v (and corresponding production period T) does not affect demand and supply of consumption goods.

We then can apply the proof for Theorem 2, because steady state consumption of all steady states with $r \in R$ is not affected, if government debt policy accommodates these changes through time of $\hat{v}$ and v .

"QED".

<u>Section 2. Waiting Period Z Depending on Life Utility</u>

In the Generalized Theorem 2 of the preceding Section 1 I assumed that the rates of change of the waiting period Z and of the period of production T are exogenously given. Yet, life expectancy is a function of the material standard of living. We can see in the demographic statistics that the average life expectancy of a national population rises with national income per head. Therefore, we can assume that the waiting period Z depends on the level of life utility U for any given moment of calendar time. This may be a problem for our <u>Assumption 8</u>: $\frac{\partial \bar{c}}{\partial \eta}\frac{1}{\bar{c}} < \frac{\partial \bar{v}}{\partial \eta}\frac{1}{\bar{v}}$ ("law of demand"), which we used to derive Theorem 2. Yet we can show that Theorem 2 still holds. With Assumption 8 we showed that for $r - g < 0$ required $\bar{c}(r; \bar{U})$ declined, as r marginally rose. From this followed that $\frac{d\bar{U}}{dr} > 0$. But then with $\frac{dZ}{d\bar{U}} > 0$ Assumption 8 holds a fortiori.

With Assumption 8 we showed that for $r - g > 0$ required $\bar{c}(r; \bar{U})$ rose, as r marginally rose. From this followed that $\frac{d\bar{U}}{dr} < 0$. If $\bar{U}$ has a positive influence on Z, and therefore on $\frac{\partial \bar{v}}{\partial \eta}\frac{1}{\bar{v}}$, then $\frac{d\bar{U}}{dr} < 0$ may destroy the inequality of Assumption 8. But this is not a problem for Theorem 2, because $\frac{d\bar{U}}{dr} < 0$ for $r > g$ is exactly what Theorem 2 proposes.

So we claim Theorem 2, even when life expectancy is pushed upwards by a higher life utility.

<u>Theorem 2- generalized: In a comparison between different steady states, each with a rate of interest remaining constant through time, life-time utility U is highest with a rate of interest equal to the rate of growth of the economic system. This remains true even if structural parameters T and Z are not necessarily constant through time; and it even remains true when life expectancy is pushed up by an enhanced life utility.</u>

Section 3. Stabilizing the Rate of Interest and Stabilizing Employment.

In Sections 1 and 2 of this Chapter, we have shown that public debt can serve as a stabilizer of the rate of interest r, if the parameters of the underlying economic models change over time. This is a stabilizing role of public debt, which is different from the Keynesian fiscal policy idea, which is supposed to stabilize employment and to dampen the business cycle. I call it the Interest-Rate-Stabilization. However, these two stabilizing roles of public debt support each other. Keynesian fiscal policy can step in, if due to a higher saving propensity aggregate demand dampens. Keynesian fiscal policy, and indeed, the automatic fiscal stabilizer contribute to the stabilization of the rate of interest.

On the other hand, interest rate stabilization helps to keep away the rate of interest from the zero lower bound, and therefore makes it easier for monetary policy to perform its part in finding the right middle between preventing unemployment and preventing inflation.

Chapter 7. The Rate of Interest as a Biased Price Signal

Section 1. Biased Price Signal in the Production System

In this Chapter, I discuss an example for a bias in the rate of interest. My general approach then also enables me to investigate the consequences of such biases. I assume that the steady state rate of interest underestimates the social rate of return on postponing consumption by some exogenously given constant parameter $\beta > 0$. Thus, the production system operates as if the rate of interest were higher by β than the actual rate of interest. The way I formalize this idea is to assume that the technique in use at a rate of interest r is $\theta(r + \beta)$, i.e. the technique, which would be used under UBPS1 with a rate of interest $r + \beta$.

It is useful to think of an economy with constant returns to scale on the macro-economic level. If UBPS1 would prevail the marginal product of capital would be equal to the steady state rate of interest . Because of the constant returns to scale assumption the "wage rate" $w\big(r; \theta(r)\big) = y - rv(r; \theta(r))$ would be the price signal of the collective marginal product of "all other factors of production, except capital".

Now, with the bias $\beta > 0$ the rate of interest is below the marginal product of capital, which in this case is $x = r + \beta$. Therefore, under constant returns to scale the "wage rate" overestimates the marginal product of "all other factors of production, except capital". Now we redefine w by means of the following equation

$$w\big(r; \theta(r+\beta)\big) = y - (r+\beta)v(r; \theta(r+\beta))$$

This definition of w no longer represents the income of the other factors of production; rather, it represents their marginal product, if we assume constant macro-economic returns to scale.

The definition of the price signal bias β implies the following

<u>Assumption 4-Beta. For any $r \in R$ we have the inequality</u>

$$\underline{w(r; \theta(r+\beta)) \geq w(r; \theta) \text{ for } \theta \in Theta}$$

Using the naming convention $\theta(r+\beta) = r + \beta$ we then obtain

$$\frac{\partial w(r; \theta(r+\beta))}{\partial \theta} = 0$$

Now we derive the Generalized Golden Rule of Accumulation. Assume, there is some $\theta^* \in Theta$, which maximizes steady state consumption $c(\theta)$ across all $\theta \in Theta$.

We have in general

$$w\big(r; \theta(r+\beta)\big) = c\big(\theta(r+\beta)\big) + gv\big(r; \theta(r+\beta)\big) - (r+\beta)v(r; \theta(r; r+\beta))$$

Partial differentiation of this equation with respect to θ results in the equation

$$0 = \frac{\partial w}{\partial \theta} = \frac{dc}{d\theta} + g\frac{\partial v}{\partial \theta} - (r+\beta)\frac{\partial v}{\partial \theta}$$

Because of $\frac{dc}{d\theta} = 0$ at $\theta = \theta^*$ and assuming $\frac{\partial v}{\partial \theta} \neq 0$ we get the result

$$r = g - \beta \text{ and } \theta^*(r+\beta) = g$$

The optimal rate of interest for the production system is the rate of growth minus the bias parameter β. In other words: if the marginal productivity of capital exceeds the rate of interest, and because the steady state welfare maximizing marginal productivity of capital equals the growth rate the welfare maximizing rate of interest is below the rate of growth.

Partial differentiation of $w(r;\theta(r+\beta))$ with respect to r, keeping θ constant yields the equation $\frac{\partial w}{\partial r} = -Tw$. Here $T > 0$ has the dimension "time". Assume one can describe the steady state economy as a set of overlapping fully vertically integrated hypothetical firms. Then T can be seen as the period of production where labor inputs and consumption good outputs get weights according to their "present values" which are calculated not by use the actual rate of interest, but by use of the "notional" rate of interest, which signifies the marginal productivity of capital.

Using this concept of the period of production T we then can partially differentiate the equation for w with respect to r, keeping θ constant

$$-Tw = \frac{\partial w}{\partial r} = g\frac{\partial v}{\partial r} - (r+\beta)\frac{\partial v}{\partial r} - v$$

For the optimal value of $r = g - \beta$ we then obtain the equation $v = wT$.

So the Böhm-Bawerk equation for the capital requirement of the production system survives even after we have allowed for a rate of interest, which possibly is a biased price signal of the marginal productivity of capital. However, we have to compute the present values of labor inputs and consumption good outputs by means of the "shadow price" of capital, i.e. by means of the marginal productivity of capital. And, as before, the Böhm-Bawerk equation only applies for the Golden Rule rate of interest.

Section 2. Compromise between the Two "Optimal" Interest Rates in the Production System and in the Consumption System

However, this is not the whole story of the optimal steady state rate of interest. We also have to consider the consumption sector. Here the Samuelson Golden Rule applies. As discussed above in Chapter 2, if the rate of interest is an unbiased price signal in the production system and in the consumption system, the optimal steady state rate of interest equals the rate of growth. If there is a bias in the production price signal then the two optima in the two systems "production" and "consumption" diverge. For the purpose of optimization, we then have to find an optimal compromise between the two sector optima.

In the following exercise, I assume that the risk free rate of interest is an unbiased price signal for the consumption sector, i.e. I assume that Assumption 6 applies (UBPS2). However, we take account of the possibility of a price signal bias β in the production sector. As I have shown in Chapter 3, for our meta-model the coefficient of intertemporal substitution allows a second degree Taylor approximation of the losses due to a deviation of the rate of interest from its optimal value. In the following, I use this approach to come to an estimate of the optimal compromise between the production sector optimum and the consumption sector optimum.

I first compute the welfare loss in the production sector, due to a deviation of r from its optimal value $g - \beta$. Let Λ_1 be the symbol for this loss. By the Taylor approximation we get

$$\Lambda_1 = \frac{1}{2} c(g - \beta; \theta(g)) \psi T^2 (r + \beta - g)^2$$

Next I compute the welfare loss in the consumption sector, due to a deviation of r from its optimal value g. Let Λ_2 be the symbol for this loss. By the Taylor approximation we get

$$\Lambda_2 = \frac{1}{2} c(g; \theta(g + \beta)) \gamma Z^2 (r - g)^2$$

Note that the loss Λ_2 depends on a different consumption level than the one for the loss Λ_1. But both consumption levels at the respective optima are determined by the production system. We then can apply the Taylor approximation for the production system again, in order to obtain the difference between the two consumption levels. We get the following

$$\begin{aligned} c\big(g; \theta(g + \beta)\big) &= c\big(g - \beta; \theta(g)\big) \left\{1 - \frac{1}{2} \psi T^2 (g + \beta - g)^2\right\} \\ &= c\big(g - \beta; \theta(g)\big) \left\{1 - \frac{1}{2} \psi T^2 \beta^2\right\} \end{aligned}$$

The equation for Λ_2 then reads

$$\Lambda_2 = c\big(g - \beta; \theta(g)\big) \left\{1 - \frac{1}{2} \psi T^2 \beta^2\right\} \frac{1}{2} \gamma Z^2 (r - g)^2$$

Let the total loss then be denoted by $\Lambda = \Lambda_1 + \Lambda_2$. We have

$$\Lambda = c\big(g - \beta; \theta(g)\big) \left[\frac{1}{2} \psi T^2 (r + \beta - g)^2 + \left\{1 - \frac{1}{2} \psi T^2 \beta^2\right\} \frac{1}{2} \gamma Z^2 (r - g)^2\right]$$

We differentiate this expression with respect to r and put the derivative equal to zero

$$\frac{d\Lambda}{dr} = c\big(g-\beta;\theta(g)\big)\left[\frac{1}{2}\psi T^2 2(r+\beta-g) + \left\{1-\frac{1}{2}\psi T^2\beta^2\right\}\frac{1}{2}\gamma Z^2 2(r-g)\right]$$
$$= 0$$

For the loss minimum we then obtain

$$\psi T^2(r+\beta-g) + \left\{1-\frac{1}{2}\psi T^2\beta^2\right\}\gamma Z^2(r-g) = 0$$

We note that for realistic values of $T\beta$ and ψ the value of $\left\{\frac{1}{2}\psi T^2\beta^2\right\}$ is quite small. For $T = 5\ years$, $\psi = 1$, and $\beta = 2\%\ p.a.$ its value is 1/200= 0.5%. Ignoring the term $\frac{1}{2}\psi T^2\beta^2$ we get the equation

$$\psi T^2(r+\beta-g) + \gamma Z^2(r-g) = 0$$

i.e. we have for the optimal compromise r^* between the two sectoral optima the equation

$$r^* = \frac{\psi T^2}{\psi T^2 + \gamma Z^2}(g-\beta) + \frac{\gamma Z^2}{\psi T^2 + \gamma Z^2}g$$

By means of this simplified equation we get the result that the optimal compromise r^* is the weighted average of the two sectoral optima where the weights are the shares of the two sectors in the second degree Taylor approximation $\Omega = \frac{1}{2}(\psi T^2 + \gamma Z^2)(r-g)^2$, which we derived in the case of UBPS1 plus UBPS2. This was Theorem 7 discussed in Chapter 3.

If we do take account of the term $\left\{\frac{1}{2}\psi T^2\beta^2\right\}$ the weight of the optimum of the production sector, $g-\beta$, rises slightly, and the weight of the optimum of the consumption sector, g, declines slightly.

In Chapter 3, Sections 7 to 9 I have developed a numerical example, which is reasonably realistic. There I used the following numbers $\psi = 1, \gamma = 1.5, T = 5\ years, Z = 10\ years$. Using the simplified equation for r^* derived above we find that the weight of the consumption sector is six-fold larger than the weight of the production sector. Thus in this example we obtain

$$r^* = \frac{1}{7}(g-\beta) + \frac{6}{7}g$$

If the bias β equals 2% p.a. then the optimal rate of interest would not be more than 30 basis points below the rate of growth of the system. (Even, if we take account of the term $\left\{\frac{1}{2}\psi T^2\beta^2\right\}$).

Chapter 8. Secular Changes in the Real Rate of Growth

In this Chapter, I take explicit account of the fact that we live in a multi-commodity world. At the same time, I allow for secular changes in the underlying real rate of growth. Indeed, the mathematical model only takes explicit account of the latter. I explain the reason.

The meta-model is characterized by the different Assumptions, which I have discussed above. It is a model with a nominal rate of growth and a nominal rate of interest. However, we are of course interested in the real rate of growth and the real rate of interest. If the underlying real rate of growth changes through time we want to change the rate of growth of the meta-model, so that its nominal rate of growth is compatible with price stability. As I mentioned earlier the well-known index problem makes it impossible to unequivocally define a rate of growth of the standard of living over a long period of time. But by periodically revising the standard commodity basket we can get reasonably good approximations of the rate of inflation over a period of several years; and thus we get a reasonably good approximation of the real rate of growth in terms of the actually consumed commodity basket.

The reason for the need to update the standard commodity basket from time to time is the structural change of demand for the different consumption goods. New goods enter the basket; some of the goods from the past disappear from the basket. Substitution and income effects due to changing relative prices and due to changes in the standard of living change the standard basket. These changes may also have an impact on the rate of growth. In addition, the average rate of technical progress may change through time.

If the (nominal) rate of growth is supposed to mirror the real rate of growth then we want to know what impact a secular change in the macro-economic real rate of growth has on the optimal rate of interest. In the following model, I investigate this question.

For this purpose, I again use the Solow production function. Assuming constant macro-economic returns to scale I can write down the value added per worker as a function of the capital intensity.

As in earlier Chapters let $f(k)$ be value added per full time worker as a function of capital intensity k. In case of a constant "natural rate of growth" g steady state consumption per worker is

$$c(k) = f(k) - gk$$

Its maximum is achieved at a rate of interest $r = f'(k)$ equal to the rate of growth (Golden Rule of Accumulation). Now I introduce a secular rate of change x of the natural rate of growth. We may think, for example, of a decline of g within a century of two percentage point, say, from 4 % p.a. to 2 % p.a.. This would mean a value of x of minus two basis points per year.

I now ask the following question: compare different "quasi-steady states" each of which is characterized by a difference $r - g$, which remains constant through time. Which is then the rate of interest of today that maximizes steady state consumption. Now, that the rate of growth changes through time, in the given steady state also the rate of interest changes through time by the same amount as the rate of growth. However, keeping up the assumption that the rate of interest is a correct price signal $r = f'(k)$ this also implies a change of k through time. Let $\dot{k}$ be this rate of change. We then have a new equation for the steady state rate of consumption

$$c(k) = f(k) - gk - \dot{k}$$

As, by assumption, the rate of interest changes through time by

$$\frac{dr}{dt} = x$$

we obtain

$$x = \frac{dr}{dt} = \frac{df'(k)}{dt} = \frac{df'(k)}{dk}\frac{dk}{dt} = f''(k)\dot{k}$$

or which is the same

$$\dot{k} = \frac{x}{f''(k)}$$

The consumption equation then reads

$$c(k) = f(k) - gk - \frac{x}{f''(k)}$$

I differentiate this expression with respect to k and put this differential equal to zero

$$0 = c'(k) = f'(k) - g + \frac{x}{[f''(k)]^2} f'''(k)$$

To get some intuition about this formula assume that $f(k)$ conforms to a constant coefficient of intertemporal substitution ψ. From the definition of ψ we get for the Solow production function

$$\psi = \frac{\partial/\frac{1}{T}}{\partial\theta} = \frac{\partial/\frac{1}{T}}{\partial k}\frac{dk}{dr} = -\frac{f(k) - kf'(k)}{k^2 f''(k)} = -\frac{1}{Tkf''(k)}$$

This implies

$$c(k) = f(k) - [g - xT\psi]k$$

In Chapter 3, Section 7 I have derived from the "Kaldor fact" of a capital-output ratio without secular trend that a constant ψ must be close to unity. And then T is a constant across different interest rates. So $xT\psi$ is a constant across different interest rates. The maximum condition for $c(k)$ then is the following

$$0 = \frac{dc}{dk} = f'(k) - [g - xT\psi]$$

or, which is the same

$$r - g = -xT\psi \approx -xT$$

For a realistic numerical evaluation let $T = 5 years$ and let $x = -2 basis\ points = -\frac{2\%\ p.a.}{100}$. It would mean that within a century the annual natural rate of growth declines by two percentage points. We then obtain for the optimal rate of interest

$$r = g + 0.10\%\ p.a.$$

As in the case of the preceding Chapter 7, one has to find the optimal compromise between the sector-optimal rate of the consumption system $r = g$ and the sector-optimal rate of the production system $r = g + 0.10\%\ p.a.$ Using again the approximation formula

$$r^* = \frac{\psi T^2}{\psi T^2 + \gamma Z^2}(g + 0.10\%\ p.a.) + \frac{\gamma Z^2}{\psi T^2 + \gamma Z^2} g$$

and, using the same weights as in the preceding Chapter leads to

$$r^* = \frac{1}{7}(g + 0.10\%) + \frac{6}{7} g = g + 0.014\%$$

The optimal steady state rate of interest exceeds the current rate of growth by 1.4 basis points.

Chapter 9. Conclusion

Here are some salient points of this paper. Assume that for demographic and other reasons the Savings Glut is here to stay: we do need "new theory" in macroeconomics. 1. Studying the steady state may be a useful tool for developing such "new theory". The constraints imposed by the condition of a steady state provide some structure. Such structure allows us to obtain results in a "meta model". The "meta-model" is defined by a set of assumptions, like, for example, the assumption that the rate of interest is a correct price signal in the context of intertemporal choice. However, many specific models fit into the meta-model. 2. We can derive certain theorems, like the Golden Rule of Accumulation, like the Böhm-Bawerk equation, like the fundamental equation of steady state capital theory $T = Z - D$.

3. We are interested to investigate the negative natural rate of interest. Therefore, we develop an alternative concept of intertemporal substitution, which can cope with negative real rates of interest. We show how our proposed parameters of intertemporal substitution ψ for the production system (and the period pf production), and γ for the consumption system (and for the "waiting period") can be used. 4. As an example we provide an approximation for the welfare losses coming from non-optimal rates of interest. 5. Here, the particularly interesting point is the importance of the Samuelson Golden Rule relative to the Phelps-Weizsäcker Golden Rule of Accumulation. 6. Another interesting point is the derivation of $\psi \approx 1$ for the Solow model by means of the fact that the capital output ratio has no secular trend.

7. Finally we show by means of a few examples how one can find insights for secular deviations from the steady state and 8. for the case that the rate of interest is a biased price signal for the marginal productivity of capital.

Annex Assumptions and Theorems

Assumption 1: The set R is a connected subset of the set of real numbers. The exogenously given growth rate g is contained in R.

Assumption 2: For any given steady state rate of interest r net investment $I(r;\theta) = gv(r;\theta)$

Assumption 3: For $r \in R$ and for $\theta \epsilon Theta$ the values $w(r;\theta)$ and $v(r;\theta)$ are continuously differentiable functions of r and θ. Addendum to Assumption 3: the variables $\overline{w}(r;\eta(r;\overline{U}))$ and $\bar{v}(r;\eta(r;\overline{U}))$ to be defined in Section 2 also are continuously differentiable functions of their arguments.

Assumption 4: (Unbiased price signal r, part 1): For each $r \in R$ the following inequality holds

$$w\big(r;\theta(r)\big) \geq w(r;\theta) \; for\ all\ \theta \in Theta$$

Assumption 4-Beta: There exists a real number β such that for each $r \in R$ the following inequality holds

$$w(r;\theta(r+\beta)) \geq w(r;\theta) \; for\ all\ \theta \in Theta$$

Assumption 5: "Law of demand". At any steady state rate of interest, $r \epsilon R$ a marginal rise in the rate of interest induces a change in the production system such that the capital-output ratio declines.

Or in a formula:

Assumption 5:

$$\frac{1}{c(\theta)}\frac{dc}{d\theta} > \frac{1}{v(r;\theta)}\frac{\partial v(r;\theta)}{\partial \theta}$$

Assumption 6: (Assumption of unbiased price signal r, part 2 (UBPS2)), Let $Eta\ (U(\bar{\eta}))\ be\ the\ set\ of\ \eta\ such\ that\ U(\eta) > U(\bar{\eta})$. Then

$$\overline{w}(r;\bar{\eta}\,) < \overline{w}(r;\eta) \;\; for\ all\ \eta \in Eta(U\ (\bar{\eta}\,))$$

Assumption 7: The level of savings compatible with maintaining the prevailing steady state rate of interest equals $g\hat{v}(r;\ \eta(r))$

Assumption 8: $\frac{\partial \bar{c}}{\partial \eta}\frac{1}{\bar{c}} < \frac{\partial \bar{v}}{\partial \eta}\frac{1}{\bar{v}}$

Assumption 9:

The partial derivative $\frac{\partial T(r;\theta)}{\partial r}$ is positive. The partial derivative $\frac{\partial Z(r;\theta)}{\partial r}$ is negative

Theorem 1: "Generalized Golden Rule of Accumulation": Under Assumptions 1 to 5 the following holds: Within the set R steady state consumption obtains its maximum at a rate of interest $r = g$. Moreover, for $r < g$ a marginal rise of the rate of interest raises steady state consumption; for $r > g$ a marginal rise of the rate of interest reduces steady state consumption.

Theorem 1 Almost Converse: Assume Assumptions 1,2,3, and 4. If, within the set R steady state consumption obtains its maximum at a rate of interest $r = g$; If moreover, for $r < g$ a marginal rise of the rate of interest raises steady state consumption; if moreover for $r > g$ a marginal rise of the rate of interest reduces steady state consumption then

$$\frac{1}{c(\theta)}\frac{dc}{d\theta} \geq \frac{1}{v(r;\theta)}\frac{\partial v(r;\theta)}{\partial \theta}$$

Theorem 2 (Golden Rule of Life Utility): Under Assumptions 1-8, comparing different steady states, life utility of the representative consumer is maximized at the steady state which exhibits a rate of interest equal to the rate of growth.

Theorem 2- generalized: In a comparison between different steady states, each with a rate of interest remaining constant through time, life-time utility U is highest with a rate of interest equal to the rate of growth of the economic system, even if structural parameters T and Z are not necessarily constant through time.

Theorem 3: At $r = g$ we have $Z - T - D = 0$

Theorem 4: With Assumptions 1-8 there is at most one steady state rate of interest, which is compatible with a zero public debt period D. If ρ is that steady state rate of interest then $D(r) > 0$ for $r > \rho$ and $D(r) < 0$ for $r < \rho$.

Theorem 5: Assumption: the steady state economy can be seen as a collection of overlapping hypothetical fully vertically integrated firms with the result that $w(r;\theta)$ is not influenced by the growth rate of the system. Then the basic duality relation of steady state capital theory holds: for any given technique θ the wage-interest curve is the same as the consumption-growth tradeoff curve applicable for the same technique.

Theorem 6A (Law of intertemporal Substitution, Part 1): Under Assumptions 1,2,3 and 4 the following holds: Keeping the weighting system r the same, a small rise in the value of θ reduces the period of production: in a formula:

$$T_2(r;\theta(r)) \leq 0$$

Theorem 6B: (Law of intertemporal Substitution, Part 2): Under Assumptions 1,3,6,and 7 the following holds: Keeping the weighting system r the same and keeping life time utility the same, a small rise in the value of η raises the value of the waiting period: in a formula:

$$Z_2(r;\eta(r)) \geq 0$$

Theorem 7A: The second order Taylor approximation of the proportional real income loss relative to the optimum on the production side is $\psi T^2(r-g)^2/2$, with ψ the coefficient of intertemporal substitution in the production system and T the "period of production", both at the optimum.

Theorem 7B: The second order Taylor approximation of the proportional real income loss relative to the optimum on the consumption side is $\gamma Z^2(r-g)^2/2$, with γ the coefficient of intertemporal substitution in the consumption system and Z the "waiting period", both at the optimum.

Theorem 7: The total proportionate loss of welfare Ω due to a steady state rate of interest different from the rate of growth , is estimated by a second order Taylor approximation to be

$$\Omega = \frac{1}{2}(\psi T^2 + \gamma Z^2)(r-g)^2$$

Theorem 8: Using the Solow model with a constant coefficient of intertemporal substitution ψ the empirical secular facts imply that ψ approximately equals unity.

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